961

THE JOURNAL

OF THE

INDIAN MATHEMATICAL SOCIETY

16 NOV 1923

EDITED BY
M. T. NARANIENGAR, M.A.

WITH THE CO-OPERATION OF

The Hon'ble R. P. PARANJPYE, M.A., D.Sc.,
Prof. A. C. L. WILKINSON, M.A., F.R.A.S.,
and others.

Volume XV, No. 4.

August 1923.

1556

MADRAS:
PRINTED BY SRINIVASA VARADACHARI & CO.

Annual Subscription Rs. 6.] [Single Copy, One Rupee.

CONTENTS.

	Page
On a General Theorem relating to the Product of two Determinants : C. Krishnamachari and M. Bhimasena Rao	73
Easy Group Theory : Prof. G. A. Miller	91
NOTES AND QUESTIONS:	
A Note in Analytical Geometry : M. T. N.	49
An Elementary Proof of Feuerbach's Theorem : K. J. Sanjana	52
Maximum inscribed Parallelogram : V. V. Sathyanarayana Murthy	53
Note on Combinatory Analysis : I. Totadri Iyengar ...	54
Solutions	56
Questions for Solution	63

A paper should contain a short and clear summary of the new results obtained and the relations in which they stand to results already known. It should be remembered that, at the present stage of mathematical research, hardly any paper is likely to be so completely original as to be independent of earlier work in the same direction; and that readers are often helped to appreciate the importance of a new investigation by seeing its connection with more familiar results.

The principal results of a paper should, when possible, be enunciated separately and explicitly in the form of definite theorems.

The Journal is open to contributions from members as well as subscribers. The Editors may also accept contributions from others.

Contributors will be supplied, if so desired, with extra copies of their contributions at net cost.

All contributions should be written legibly on one side only of the paper, and all diagrams should be neatly and accurately drawn on separate slips.

All communications intended for the Journal should be addressed to the Hony. Joint Secretary and Editor M. T. NARANIENGAR, M.A., 24, Mallesvaram, Bangalore, or Joint Editor K. ANANDA RAO, M.A., Professor, Presidency College, Madras.

All business communications regarding the Journal should be addressed to the Hony. Asst. Secy. M. RAGHUNATHA AIYANGAR, M.A., Presidency College, Triplicane, Madras.

Enquiries by intending members of the Society and all other communications may be addressed to the Hony. Joint Secretary P. V. SESHU AIYAR, B.A., L.T., 18, Pycroft's Road, Triplicane, Madras.

On a General Theorem relating to the Product of two Determinants.

By C. Krishnamachari and M. Bhimasena Rao.

[References to the authors' earlier papers :—

I. Determinants whose elements are Eulerian, Prepared Bernoullian and other numbers: *Journal of the Indian Mathematical Society*, Vol. XIV, No. 2.

II. Do. : *J. I. M. S.* Vol. XIV, No. 4.

III, Contributions to the Evaluation of General Persymmetric Determinants: Communicated to the *London Mathematical Society* through Mr. G. H. Hardy, M.A., F.R.S., August 1922.

IV, On Certain Persymmetric Determinants: *J. I. M. S.* Vol. XIV, No. 6.

These papers will be referred to here as I, II, III, IV.]

[We have contributed in a series of papers our researches relating to the evaluation of certain classes of determinants to the *Journal of the Indian Mathematical Society*. The main results relating to persymmetrics have been communicated to the *London Mathematical Society*. In the course of our work, we have discovered a very elegant theorem, which has many interesting applications. In particular, we find that most of the results proved in our previous papers can be deduced from this theorem. This paper contains this theorem and some of its applications].

§ 1. Consider the following system of n equations in $(m + n)$ variables. We may for the sake of definiteness suppose $n > m$.

$$\left.\begin{array}{l} A_{1,1}\, x_1 + A_{1,2}\, x_2 + \dots \quad + A_{1,m+n}\, x_{m+n} = 0. \\ A_{2,1}\, x_1 + A_{2,2}\, x_2 + \dots \quad + A_{2,m+n}\, x_{m+n} = 0. \\ \dots \qquad \dots \qquad \dots \\ A_{n,1}\, x_1 + A_{n,2}\, x_2 + \dots \quad + A_{n,m+n}\, x_{m+n} = 0. \end{array}\right\} \qquad (1)$$

Let there be a set of m distinct solutions to the above, *viz.*,

$$\left.\begin{array}{l} B_{11}, B_{12}, \ldots\ldots\ldots B_{1, m+n} \\ B_{21}, B_{22}, \ldots\ldots\ldots B_{2, m+n} \\ \ldots \quad \ldots \quad \ldots \\ B_{m,1}, B_{m,2}, \ldots\ldots B_{m, m+n}. \end{array}\right\} \qquad (2)$$

Consider the two matrices A and B of n and m rows respectively, both of $(m + n)$ columns ;

$$A \equiv \left\{\begin{array}{l} A_{11}\ A_{12} \ldots\ldots\ldots A_{1, m+n} \\ A_{21}\ A_{22} \ldots\ldots\ldots A_{2, m+n} \\ \ldots \quad \ldots \quad \ldots \\ A_{n,1}\ A_{n,2} \ldots\ldots\ldots A_{n, m+n} \end{array}\right\} ; B \equiv \left\{\begin{array}{l} B_{11}\ B_{12} \ldots\ldots\ldots B_{1, m+n} \\ B_{21}\ B_{22} \ldots\ldots\ldots B_{2, m+n} \\ \ldots \quad \ldots \quad \ldots \\ B_{m1}\ B_{m2} \ldots\ldots B_{m, m+n} \end{array}\right\}. \qquad (3)$$

Let D stand for the determinant of $(m + n)$ rows and columns obtained by placing B below A.

The matrices will be referred to as the A matrix (of co-efficients) and the B matrix (of solutions) of the equations (1). Then the following theorem holds.

THEOREM I.

Let Δ_1 denote the determinant formed by taking the first n columns from the matrix A ; Δ_2 the determinant formed by taking the first m columns from the matrix B. Let Δ_1' and Δ_2' denote the corresponding determinants formed from the last n and m columns of the same matrices. Then $\Delta_1 . \Delta_2 = (-1)^{mn}\, \Delta_1' . \Delta_2'$. ... ... (4)

Proof: Convert each of the above matrices A and B into a determinant of $(m + n)$ rows and columns by introducing 1 along a diagonal line in such a way that their product is equal to $\Delta_1 . \Delta_2$. Multiply the two determinants of the $(m + n)$th order by the ordinary rule for the multiplication of determinants, row by row and apply (1). We have :

195317

$$\Delta_1 . \Delta_2 = \begin{vmatrix} \cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots, 1, 0, 0 \\ \cdots\cdots\cdots\; 0, 1, 0 \\ \cdots\cdots\cdots\; 0, 0, 1 \\ A_{11}, A_{12}, \ldots A_{1,\, m+n} \\ A_{21}, A_{22}, \ldots A_{2,\, m+n} \\ \cdots\cdots\cdots\cdots \\ A_{n1}, A_{n2}, \ldots A_{n,\, m+n} \end{vmatrix} \times \begin{vmatrix} \cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots, 1, 0, 0 \\ \cdots\cdots\cdots\; 0, 1, 0 \\ \cdots\cdots\cdots\; 0, 0, 1 \\ B_{11}, B_{12}, \ldots B_{1,\, m+n} \\ B_{21}, B_{22}, \ldots B_{2,\, m+n} \\ \cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots \\ B_{m1}, B_{m2}, \ldots B_{m,\, m+n} \end{vmatrix}$$

$$= \begin{vmatrix} 0, 0, \cdots\cdots 0, B_{1,\, n+1}, B_{2,\, n+1} \cdots\cdots B_{m,\, n+1} \\ 0, 0, \cdots\cdots 0, B_{1,\, n+2}, B_{2,\, n+2} \cdots\cdots B_{m,\, n+2} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 0, 0, \cdots\cdots 0, B_{1,\, m+n}, B_{2,\, m+n} \cdots\cdots B_{m,\, m+n} \\ A_{1,\, m+1}\; A_{1,\, m+2} \cdots\cdots A_{1,\, m+n}, \cdots\cdots 0, 0, \cdots\cdots 0 \\ A_{2,\, m+1}\; A_{2,\, m+2} \cdots\cdots A_{2,\, m+n}, \cdots\cdots 0, 0, \cdots\cdots 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ A_{n,\, m+1}\; A_{n,\, m+2} \cdots\cdots A_{n,\, m+n}, \cdots\cdots 0, 0, \cdots\cdots 0 \end{vmatrix}$$

$$= \Delta_1' . \Delta_2'$$

§ 2. The last theorem may be generalized thus:

THEOREM II.

Let any determinant Δ_1 of n rows and columns be formed from the matrix A (of coefficients) and any determinant Δ_2 of m rows and columns be formed from the matrix B (of solutions). Let Δ'_1 be the determinant formed from A by omitting from it the columns corresponding to those taken in Δ_2; and let Δ'_2 be the determinant formed from B by omitting from it the columns corresponding to those taken in Δ_1. Then

$$\Delta_1 \, \Delta_2 = \Delta'_1 \, \Delta'_2 \, (-1)^{mn+h+k} \quad \ldots \quad \ldots \quad (5)$$

where h is the number of changes of sign introduced in making the columns in Δ_1 the first consecutive columns in D and k is the number of changes of sign in making the columns of Δ_2 the first consecutive columns in D.

The proof is similar to that of Theorem I. For the sake of definiteness, take a system of 3 solutions for 5 equations in 8 unknowns.

$$A \equiv \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{18} \\ a_{21} & a_{22} & \cdots & a_{28} \\ \cdots & \cdots & \cdots & \cdots \\ a_{51} & a_{52} & \cdots & a_{58} \end{vmatrix} \; ; \; B \equiv \begin{vmatrix} b_{11} & b_{12} & \cdots & b_{18} \\ b_{21} & b_{22} & \cdots & b_{28} \\ b_{31} & b_{32} & \cdots & b_{38} \end{vmatrix} .$$

We shall prove that

$$\begin{vmatrix} a_{11} & a_{13} & a_{14} & a_{17} & a_{18} \\ a_{21} & a_{23} & a_{24} & a_{27} & a_{28} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{51} & a_{53} & a_{54} & a_{57} & a_{58} \end{vmatrix} \times \begin{vmatrix} b_{12} & b_{13} & b_{18} \\ b_{22} & b_{23} & b_{28} \\ b_{32} & b_{33} & b_{38} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & a_{14} & a_{15} & a_{16} & a_{17} \\ a_{21} & a_{24} & a_{25} & a_{26} & a_{27} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{51} & a_{54} & a_{55} & a_{56} & a_{57} \end{vmatrix} \times \begin{vmatrix} b_{12} & b_{15} & b_{16} \\ b_{22} & b_{25} & b_{26} \\ b_{32} & b_{35} & b_{36} \end{vmatrix}, \quad \ldots \quad (6)$$

since $n=5$, $m=3$, $h=8$, $k=7$.

For, the left-hand side is equal to the product of the determinants

$$\begin{vmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} & a_{27} & a_{28} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} & a_{57} & a_{58} \end{vmatrix}$$

and

$$\begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ b_{11} & b_{12} & b_{13} & b_{14} & b_{15} & b_{16} & b_{17} & b_{18} \\ b_{21} & b_{22} & b_{23} & b_{24} & b_{25} & b_{26} & b_{27} & b_{28} \\ b_{31} & b_{32} & b_{33} & b_{34} & b_{35} & b_{36} & b_{37} & b_{38} \end{vmatrix};$$

and is thus written in the form

$$= - \begin{vmatrix} 0 & 0 & 0 & 0 & 0 & b_{12} & b_{22} & b_{32} \\ 0 & 0 & 0 & 0 & 0 & b_{15} & b_{25} & b_{35} \\ 0 & 0 & 0 & 0 & 0 & b_{16} & b_{26} & b_{36} \\ a_{11} & a_{14} & a_{15} & a_{16} & a_{17} & 0 & 0 & 0 \\ a_{21} & a_{24} & a_{25} & a_{26} & a_{27} & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{51} & a_{54} & a_{55} & a_{56} & a_{57} & 0 & 0 & 0 \end{vmatrix}$$

= Expression on the right in (6)

Ex. 1. Let $(x_1\ y_1\ z_1)$ be a solution of

$$a_1x + b_1y + c_1z = 0$$
$$a_2x + b_2y + c_2z = 0.$$

$$A \equiv \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix};\ B \equiv [x_1\ y_1\ z_1];$$

We get
$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} x_1 = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} z_1.$$

$$i.e.\ \frac{x_1}{(b_1\ c_2)} = \frac{z_1}{(a_1\ b_2)}.$$

Ex. 2. Let $(x_1\ y_1\ z_1)$ be the solution of the equations

$$a_rx + b_ry + c_rz + d_r = 0, \qquad [r = 1, 2, 3].$$

$$A \equiv \begin{Bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{Bmatrix};\ B \equiv [x_1\ y_1\ z_1\ 1].$$

We get $(a_1b_2c_3)x_1 = -(b_1c_2d_3)$. $\therefore\ x_1 = -\dfrac{(b_1c_2d_3)}{(a_1b_2c_3)}$.

Ex. 3. Let a determinant of the nth order be divided into two rectangular arrays of μ and v rows respectively so that $\mu + v = n$, and let μv sums of products be formed by operating with one array on the other as in the multiplication of determinants. If then such relations exist among the constituents that all these sums of products vanish separately, the determinants of order μ formed from the first array are proportional to the determinants of order v formed from the complementary constituents of the second.

[Cf. the proof in Burnside and Panton: *Theory of Equations*, Vol. II, p. 67, Ex. 44.].

This important theorem follows easily from Theorem II above.

Let us take a determinant of the fifth order, and divide it into 2 arrays of 3 and 2 rows respectively.

$$\Delta \equiv \begin{vmatrix} a_1 & a_2 & \cdots\cdots & a_5 \\ b_1 & b_2 & \cdots\cdots & b_5 \\ c_1 & c_2 & \cdots\cdots & c_5 \\ x_1 & x_2 & \cdots\cdots & x_5 \\ y_1 & y_2 & \cdots\cdots & y_5 \end{vmatrix};\ A \equiv \begin{Bmatrix} a_1 & a_2 & \cdots\cdots & a_5 \\ b_1 & b_2 & \cdots\cdots & b_5 \\ c_1 & c_2 & \cdots\cdots & c_5 \end{Bmatrix};B \equiv \begin{Bmatrix} x_1 & x_2 & \cdots\cdots & x_5 \\ y_1 & y_2 & \cdots\cdots & y_5 \end{Bmatrix};$$

and let
$$a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 + a_5x_5 = 0,$$
$$\Sigma\, b_1x_1 = 0,\ \Sigma\, c_1x_1 = 0,$$
$$\Sigma\, a_1y_1 = 0,\ \Sigma\, b_1y_1 = 0,\ \Sigma\, c_1y_1 = 0$$
so that the conditions of Theorem II above are fulfilled.

Applying the theorem, we have

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} = \begin{vmatrix} a_3 & a_4 & a_5 \\ b_3 & b_4 & b_5 \\ c_3 & c_4 & c_5 \end{vmatrix} \begin{vmatrix} x_4 & x_5 \\ y_4 & y_5 \end{vmatrix}$$

i.e.,
$$\frac{(a_1\, b_2\, c_3)}{(x_4\, y_5)} = \frac{(a_3\, b_4\, c_5)}{(x_1\, y_2)} = \text{etc.}$$

§ 3. We now proceed to apply the above theorems to obtain some interesting results relating to determinants proved by other (direct) methods in our previous papers referred to above.

From Paper I, we take the following equations. E_n is the nth Eulerian number (being the co-efficient of $\dfrac{x^{2n}}{2n\,!}$ in the expansion of sec x in ascending powers of x) ; and
$$S_{n,r} = \Sigma(2n-1)^r\ \Sigma(2n-3)^r \ldots\ldots \Sigma(2n-2r+1)^r,$$
a number obtained by a method of multiple summation as explained in § 1, of that paper. (See Appendix I).

$$\left.\begin{aligned} &E_o\,.\,S_{n,n} - E_1\,.\,S_{n,n-1} + E_2\,.\,S_{n,n-2} - \ldots \\ &\qquad + (-1)^{n-1} E_{n-1}\,.\,S_{n,1} + (-1)^n E_n = 0, \\ &E_1\,.\,S_{n,n} - E_2\,.\,S_{n,n-1} + E_3\,.\,S_{n,n-2} - \ldots \\ &\qquad + (-1)^{n-1} E_n\,.\,S_{n,1} + (-1)^n E_{n+1} = 0, \\ &\ldots \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots \\ &E_{n-1}\,.\,S_{n,n} - E_n\,.\,S_{n,n-1} + E_{n+1}\,.\,S_{n,n-2} - \ldots \\ &\qquad + (-1)^{n-1} E_{2n-2}\,.\,S_{n,1} + (-1)^n E_{2n-1} = 0, \\ &E_n\,.\,S_{n,n} - E_{n+1}\,.\,S_{n,n-1} + E_{n+2}\,.\,S_{n,n-2} - \ldots \\ &\qquad + (-1)^{n-1} E_{2n-1}\,.\,S_{n,1} + (-1)^n E_{2n} = (2n\,!)^2. \end{aligned}\right\} \quad (7)$$

We have obviously the persymmetric

$$[E_o, E_r] = \begin{vmatrix} E_o & E_1 & \cdots & E_r \\ E_1 & E_2 & \cdots & E_{r+1} \\ \cdots & \cdots & \cdots \\ E_r & E_{r+1} & \cdots & E_{2r} \end{vmatrix} = (2r\,!)^2 (2r-2\,!)^2 (2r-4\,!)^2 \ldots (4\,!^2)(2\,!^2).$$

From (7), we write the following A and B matrices, satisfying the conditions of our theorems:—

$$A \equiv \left\{ \begin{matrix} E_o & E_1 & \ldots & \ldots & E_n \\ E_1 & E_2 & \ldots & \ldots & E_{n+1} \\ & \ldots & \ldots & \ldots & \\ E_r & E_{r+1} & \ldots & \ldots & E_{n+r} \end{matrix} \right\};$$

$$B \equiv \left\{ \begin{matrix} S_{n,n} & -S_{n,n-1} & S_{n,n-2} & \ldots & & (-1)^{n-1} S_{n,1} & (-1)_n \\ S_{n-1,n-1} & -S_{n-1,n-2} & S_{n-1,n-3} & \ldots & & (-1)^{n-1} & 0 \\ S_{n-2,n-2} & -S_{n-2,n-3} & \ldots & \ldots & (-1)^{n-2} & 0 & 0 \\ \ldots & \ldots & & \ldots & & \ldots & \ldots \\ S_{r+1,r+1} & -S_{r+1,r} & S_{r+1,r-1} & & 0 & 0 & 0 \end{matrix} \right\}.$$

A is a matrix of $n+1$ columns and $r+1$ rows, and B of $n+1$ columns and $n-r$ rows.

Let Δ be the determinant of $(n-r)$ rows and columns obtained from the last $n-r$ columns in B. Then

$$\Delta = (-1)^{n(n-r)}$$

since all the elements along a diagonal are unity and all the elements on one side of it are zeros.

Applying Theorem II, we have

$$(-1)^{n(n-r)} [E_{n-r}, E_n] = (-1)^{(r+1)(n-r)}$$

$$\times \left| \begin{matrix} S_{n,n} & -S_{n,n-1} & S_{n,n-2} & \ldots & (-1)^{n-r-1} S_{n,r+1} \\ S_{n-1,n-1} & -S_{n-1,n-2} & S_{n-1,n-3} & \ldots & (-1)^{n-r-1} S_{n-1,r} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ S_{r+1,r+1} & -S_{r+1,r} & S_{r+1,r-1} & \ldots & \ldots \end{matrix} \right| [E_0, E_r]$$

it being remembered that $S_{r,0} = 1$ and $S_{r,-k} = 0$, where k is any positive integer,

i.e.,

$$[E_{n-r}, E_n] = (2r!)^2 (2r-2!)^2 \ldots (2!)^2 \times \left| \begin{matrix} S_{n,r+1} & S_{n,r+2} & S_{n,n} \\ S_{n-1,r} & S_{n-1,r+1} & S_{n-1,n-1} \\ \ldots & \ldots & \ldots \\ \ldots & \ldots & S_{r+1,r+1} \end{matrix} \right| \quad \ldots (8)$$

§ 3.1. Let $k_{n,r} = \Sigma(2n)^2 \, \Sigma(2n-2)^2 \ldots \Sigma(2n-2r+2)^2$. From the following equations taken from I,.

$$\left.\begin{aligned}
&E_1 k_{n,n} - E_2 k_{n,n-1} + E_3 k_{n,n-2} - \cdots \\
&\qquad + (-1)^{n-1} E_n \; k_{n,1} + (-1)^n E_{n+1} = 0, \\
&E_2 k_{n,n} - E_3 k_{n,n-1} + E_4 k_{n,n-2} - \cdots \\
&\qquad + (-1)^{n-1} E_{n-1} k_{n,1} + (-1)^n E_{n+2} = 0, \\
&\qquad \cdots \qquad \cdots \qquad \cdots \\
&E_n k_{n,n} - E_{n+1} k_{n,n-1} + E_{n+2} k_{n,n-2} - \cdots \\
&\qquad + (-1)^{n-1} E_{2n-1} k_{n,1} + (-1)^n E_{2n} = 0, \\
&E_{n+1} k_{n,n} - E_{n+2} \; k_{n,n-1} + \quad \cdots \qquad \cdots \\
&\qquad + (-1)^{n-1} E_{2n} k_{n,1} + (-1)^n E_{2n-1} = (2n+1!)^2.
\end{aligned}\right\} \quad \ldots (9)$$

[From (7) or (9), we find the value of the persymmetric $[E_1, E_r]$ to be $(2r-1!)^2 \, (2r-3!)^2 \ldots\ldots (1!)^2$].

$$A \equiv \left\{\begin{matrix} E_1 & E_2 & \cdots & E_{n+1} \\ E_2 & E_3 & \cdots & E_{n+2} \\ & \cdots & \cdots & \cdots \\ E_r & E_{r+1} & \cdots & E_{n+r} \end{matrix}\right\};$$

$$B \equiv \left\{\begin{matrix} k_{n,n} & -k_{n,n-1} & k_{n,n-2} \cdots & (-1)^{n-1} k_{n,1} & (-1)^n \\ k_{n-1,n-1} & -k_{n-1,n-2} & \cdots & (-1)^{n-1} & 0 \\ \cdots & \cdots & & \cdots & \cdots \\ k_{r,r} & -k_{r,r-1} & k_{r,r-2} & & \end{matrix}\right\}.$$

By applying Theorem II, we find after some simple algebra,

$$[E_{n-r}, E_n] = [E_1, E_{r+1}] \begin{vmatrix} k_{n-1,r+1} & k_{n-1,r+2} & \cdots & k_{n-1,n-1} \\ k_{n-2,r} & k_{n-2,r+1} & \cdots & k_{n-2,n-2} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & k_{r+1,r} & k_{r+1,r+1} \end{vmatrix} \quad \ldots (10)$$

The expression in (10) can also be obtained directly from (8), and the determinant on the right in (10) is of order one less than that in (8).

Ex. Write $r = 2$, and $n = 5$.

$$\begin{vmatrix} E_3 & E_4 & E_5 \\ E_4 & E_5 & E_6 \\ E_5 & E_6 & E_7 \end{vmatrix} = \begin{vmatrix} 61, & 1,385 & 50,521 \\ 1,385 & 50,521 & 2,702,765 \\ 50,521 & 2,702,765 & 199,360,981 \end{vmatrix}$$

$$= 35,651,552,544,000.$$

$$= 2^2 . 24^2 . 225 . (68,772,285).$$

and

$$\begin{vmatrix} k_{4,3} & k_{4,4} \\ k_{3,2} & k_{3,3} \end{vmatrix} = \begin{vmatrix} \Sigma 8^2 \Sigma 6^2 \Sigma 4^2 & \Sigma 8^2 \Sigma 6^2 \Sigma 4^2 \Sigma 2^2 \\ \Sigma 6^2 \Sigma 4^2 & \Sigma 6^2 \Sigma 4^2 \Sigma 2^2 \end{vmatrix}$$

$$= \begin{vmatrix} 122156 & 230481 \\ 1519 & 3429 \end{vmatrix}$$

$$= 68,772,285.$$

In a similar manner, we may obtain the values of similar determinants involving $b_n = 2^{-n}(2^{2n} - 1)\frac{B_n}{2n}$, when B_n is the nth Bernoullian number.

§ 4. We next proceed to obtain some very general theorems.

Let a_n be any function of the integral variable n; and let

$$A_r = \Sigma a_1 \, \Sigma a_2 \, \ldots\ldots \, \Sigma a_r,$$

and

$${}_{2n-1}\alpha_{2r-1} = \Sigma a_{2n-1} \, \Sigma a_{2n-3} \, \ldots\ldots \, \Sigma a_{2r-1}$$

The number A_r is very general, and various co-efficients are particular cases of A_r. (See Paper II). In II we have established the following equations :—

$$\left.\begin{array}{l} A_0 . {}_{2n-1}\alpha_1 - A_1 . {}_{2n-1}\alpha_3 + A_2 . {}_{2n-1}\alpha_5 - \cdots \\ \quad + (-1)^{n-1} {}_{2n-1}\alpha_{2n-1} . A_{n-1} + (-1)^n A_n = 0. \\ A_1 . {}_{2n-1}\alpha_1 - A_2 . {}_{2n-1}\alpha_3 + A_3 . {}_{2n-1}\alpha_5 - \cdots \\ \quad + (-1)^{n-1} {}_{2n-1}\alpha_{2n-1} . A_n + (-1)^n A_{n+1} = 0. \\ \quad \cdots \qquad \cdots \qquad \cdots \qquad \cdots \\ A_{n-1} . {}_{2n-1}\alpha_1 - A_n . {}_{2n-1}\alpha_3 + A_{n+1} . {}_{2n-1}\alpha_5 - \cdots \\ \quad + (-1)^{n-1} {}_{2n-1}\alpha_{2n-1} . A_{2n-2} + (-1)^n A_{2n-1} = 0. \\ A_n . {}_{2n-1}\alpha_1 - A_{n+1} . {}_{2n-1}\alpha_3 + A_{n+2} . {}_{2n-1}\alpha_5 - \cdots \\ \quad + (-1)^{n-1} {}_{2n-1}\alpha_{2n-1} . A_{2n-1} + (-1)^n A_{2n} = a_1 a_2 \cdots a_{2n}. \end{array}\right\} \quad (11)$$

and

$$\left.\begin{aligned}
&A_1 \,.\, {}_{2n}\alpha_2 - A_2 \,.\, {}_{2n}\alpha_4 + \dots \\
&\quad + (-1)^{n-1}\, {}_{2n}\alpha_{2n} \,.\, A_n + (-1)^n A_{n+1} = 0. \\
&A_2 \,.\, {}_{2n}\alpha_2 - A_3 \,.\, {}_{2n}\alpha_4 + \dots \\
&\quad + (-1)^{n-1}\, {}_{2n}\alpha_{2n} \,.\, A^{n+1} + (-1)^n A_{n+2} = 0. \\
&\dots \qquad \dots \qquad \dots \qquad \dots \\
&A_n \,.\, {}_{2n}\alpha_2 - A_{n+1} \,.\, {}_{2n}\alpha_4 + \dots \\
&\quad + (-1)^{n-1}\, {}_{2n}\alpha_{2n} \,.\, A_{2n-1} + (-1)^n A_{2n} = 0. \\
&A_{n+1} \,.\, {}_{2n}\alpha_2 - A_{n+2} \,.\, {}_{2n}\alpha_4 + \dots \\
&\quad + (-1)^{n-1}\, {}_{2n}\alpha_{2n} \,.\, A_{2n} + (-1)^n A_{2n+1} = a_1 a_2 \dots a_{2n-1}.
\end{aligned}\right\} \quad (12)$$

Proceeding as in § 3 above, we find $[A_{n-r}, A_n]$

$$= [A_o, A_r] \begin{vmatrix} {}_{2n-1}\alpha_{2n-2r-1}, & {}_{2n-1}\alpha_{2n-2r-3} & \dots & {}_{2n-1}\alpha_3, & {}_{2n-1}\alpha_1 \\ {}_{2n-3}\alpha_{2n-2r-1}, & {}_{2n-3}\alpha_{2n-2r-3} & \dots & {}_{2n-3}\alpha_3, & {}_{2n-3}\alpha_1 \\ \dots & \dots & \dots & \dots & \\ \dots & \dots & \dots & {}_{2r+1}\alpha_3. & {}_{2r+1}\alpha_1 \end{vmatrix} \quad \dots (13)$$

$$= [A_1, A_{r+1}] \begin{vmatrix} {}_{2n-2}\alpha_{2n-2r-2}, & {}_{2n-2}\alpha_{2n-2r-4} & \dots & {}_{2n-2}\alpha_4, & {}_{2n-2}\alpha_2 \\ {}_{2n-4}\alpha_{2n-2r-2}, & {}_{2n-4}\alpha_{2n-2r-4} & \dots & {}_{2n-4}\alpha_4, & {}_{2n-4}\alpha_2 \\ \dots & \dots & \dots & \dots & \\ \dots & \dots & \dots & {}_{2r+2}\alpha_4, & {}_{2r+2}\alpha_2 \end{vmatrix} \quad \dots (14)$$

and from (11) and (12), we find immediately

$$[A_o, A_r] = (a_1 a_2)^r (a_3 a_4)^{r-1} \quad \dots \quad (a_{2r-1}\, a_{2r})$$

$$[A_1, A_{r+1}] = a_1^{\,r+1} (a_2 a_3)^r (a_4 a_5)^{r-1} \dots \quad (a_{2r}\, a_{2r+1}).$$

In (13), and (14), if the right suffix of an α is greater by 2 than the left, the value of the α is to be put equal to unity, and if it is greater by more than 2 its value is to be put equal to zero.

§ 5. It is easily seen, that, if q_n is the denominator of the nth convergent of the continued fraction

$$\frac{1}{x-}\,\frac{a_1}{x-}\,\frac{a_2}{x-}\dots\dots$$

$$\begin{aligned} q_{2n} &= x^{2n} - {}_{2n-1}\alpha_{2n-1} \,.\, x^{2n-2} + {}_{2n-1}\alpha_{2n-3} \,.\, x^{2n-4} - \dots \\ &\quad + (-1)^n\, {}_{2n-1}\alpha_1 \qquad \dots \qquad \dots (15) \end{aligned}$$

$$\begin{aligned} q_{2n+1} &= x^{2n+1} - {}_{2n}\alpha_{2n} \,.\, x^{2n-1} + {}_{2n}\alpha_{2n-2} \,.\, x^{2n-3} - \dots \\ &\quad + (-1)^n\, {}_{2n}\alpha_2 \,.\, x \qquad \dots \qquad \dots (16) \end{aligned}$$

From (11) and (15), we write down the following two matrices satisfying the conditions of our theorems:

$$A \equiv \left\{ \begin{matrix} 1 & x^2 & x^4 & \ldots & \ldots & x^{2n} & (-1)^{n+1} \\ A_0 & A_1 & A_2 & \ldots & \ldots & A_n & 0 \\ A_1 & A_2 & A_3 & \ldots & \ldots & A_{n+1} & 0 \\ & \ldots & \ldots & & \ldots & \ldots & \\ A_{r-1} & A_1 & A_{r+1} & \ldots & \ldots & A_{n+r-1} & 0 \end{matrix} \right\}$$

$$B \equiv \left\{ \begin{matrix} {}_{2n-1}\alpha_1 & (-)_{2n-1}\alpha_3 & {}_{2n-1}\alpha_5 & \ldots & (-1)^{n-1}\,{}_{2n-1}\alpha_{2n-1} & (-1)^n q_{2n} \\ {}_{2n-3}\alpha_1 & (-)_{2n-3}\alpha_3 & {}_{2n-3}\alpha_5 & \ldots & (-1)^{n-1} \quad 0 & -q_{2n-2} \\ {}_{2n-5}\alpha_1 & (-)_{2n-5}\alpha_3 & {}_{2n-5}\alpha_5 & \ldots & 0 \quad 0 & +q_{2n-4} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ {}_{2r-1}\alpha_1 & (-)_{2r-1}\alpha_3 & {}_{2r-1}\alpha_5 & \ldots & 0 \quad 0 & (-)^{n-r} q_{2r} \end{matrix} \right\}$$

If Δ is the determinant of $n-r+1$ rows and columns obtained by taking $n-r+1$ columns consecutively from the end in B leaving aside the last,

$$\Delta = (-1)^{n(n-r+1)}.$$

Hence,

$$\begin{vmatrix} x^{2n-2r} & x^{2n-2r+2} & \ldots & x^{2n} \\ A_{n-r} & A_{n-r+1} & \ldots & A_n \\ A_{n-r+1} & A_{n-r+2} & \ldots & A_{n+1} \\ \ldots & \ldots & & \\ A_{n-1} & A_n & \ldots & A_{n+r-1} \end{vmatrix} \Delta = (-1)^{r(n-r+1)} \begin{vmatrix} 1 & x^2 & x^4 & \ldots & x^{2r-2} & (-1)^{n+1} \\ A_0 & A_1 & A_2 & \ldots & A_{r-1} & 0 \\ A_1 & A_2 & A_3 & \ldots & A_r & 0 \\ & \ldots & & & & \ldots \\ A_{r-1} & A_r & A_{r+1} & \ldots & A_{2r-2} & 0 \end{vmatrix}$$

$$\times \begin{vmatrix} {}_{2n-1}\alpha_1 & -{}_{2n-1}\alpha_3 & \ldots & (-1)^{n-r-1}\,{}_{2n-1}\alpha_{2n-2r-1} & q_{2n} \\ {}_{2n-3}\alpha_1 & -{}_{2n-1}\alpha_5 & \ldots & (-1)^{n-r-1}\,{}_{2n-3}\alpha_{2n-2r-1} & -q_{2n-2} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ {}_{2r-1}\alpha_1 & -{}_{2r-1}\alpha_3 & \ldots & \ldots & (-1)^{n-r} q_{2r} \end{vmatrix}$$

which after simplification gives

$$\begin{vmatrix} x^{2n-2r} & x^{2n-2r+2} & \ldots & x^{2n} \\ A_{n-r} & A_{n-r+1} & \ldots & A_n \\ A_{n-r+1} & A_{n-r+2} & \ldots & A_{n+1} \\ \ldots & \ldots & & \ldots \\ A_{n-1} & A_n & \ldots & A_{n+r-1} \end{vmatrix} = (-1)^{\frac{(n-r)(n-r+1)}{2}+n} [A_0, A_{r-1}].T$$

where

$$T \equiv \begin{vmatrix} q_{2n} & {}_{2n-1}\alpha_1 & {}_{2n-1}\alpha_3 & \ldots & {}_{2n-1}\alpha_{2n-2r-1} \\ -q_{2n-2} & {}_{2n-3}\alpha_1 & {}_{2n-3}\alpha_3 & \ldots & {}_{2n-3}\alpha_{2n-2r-1} \\ \ldots & \ldots & \ldots & & \ldots \\ (-1)^{n-r} q_{2r} & {}_{2r-1}\alpha_1 & {}_{2r-1}\alpha_3 & \ldots & \ldots \end{vmatrix} \quad \ldots (17)$$

In a similar manner, from (12) and (16), we find that

$$\begin{vmatrix} x^{2n-2r+1} & x^{2n-2r+3} & \ldots & x^{2n+1} \\ A_{n-r+1} & A_{n-r+2} & \ldots & A_{n+1} \\ A_{n-r+2} & A_{n-r+3} & \ldots & A_{n+2} \\ \ldots & \ldots & & \ldots \\ A_n & A_{n+1} & \ldots & A_{n+r} \end{vmatrix} = (-1)^{\frac{(n-r)(n-r+1)}{2}+n} [A_1, A_r]$$

$$\times \begin{vmatrix} q_{2n+1} & {}_{2n}\alpha_2 & {}_{2n}\alpha_4 & \ldots & {}_{2n}\alpha_{2n-2r} \\ -q_{2n-1} & {}_{2n-2}\alpha_2 & {}_{2n-2}\alpha_4 & \ldots & {}_{2n-2}\alpha_{2n-2r} \\ \ldots & \ldots & \ldots & & \ldots \\ (-1)^{n-r} q_{2r+1} & {}_{2r}\alpha_2 & {}_{2r}\alpha_4 & & \ldots \end{vmatrix} \quad \ldots (18)$$

As particular cases of (17) and (18), writing $r = n$, we obtain

$$\begin{vmatrix} 1 & x^2 & x^4 & \ldots & x^{2n} \\ A_o & A_1 & A_2 & \ldots & A_n \\ A_1 & A_2 & A_3 & \ldots & A_{n+1} \\ \ldots & & \ldots & & \ldots \\ A_{n-1} & A_n & A_{n+1} & \ldots & A_{2n-1} \end{vmatrix} = (-1)^n [A_o, A_{n-1}] \, q_{2n}(x) \quad \ldots (19)$$

$$\begin{vmatrix} x & x^3 & \ldots & x^{2n+1} \\ A_1 & A_2 & \ldots & A_{n+1} \\ A_2 & A_3 & \ldots & A_{n+2} \\ \ldots & \ldots & & \ldots \\ A_n & A_{n+1} & \ldots & A_{2n} \end{vmatrix} = (-1)^n [A_1, A_n] \, q_{2n+1}(x) \quad \ldots (20)$$

§ 6. Consider the following matrices which are easily seen to satisfy the conditions of our theorems:—

$$A \equiv \begin{Bmatrix} x_1 & x_1^3 & \dots & x_1^{2n-1} & \dots & x_1^{2n+2r-1} & (-1)^{n+r} & 0 & 0 \dots & \\ x_2 & x_2^3 & \dots & x_2^{2n-1} & \dots & x_2^{2n+2r-1} & 0 & (-1)^{n+r-1} & 0 \dots & \\ & \dots & & \dots & & \dots & & \dots & \dots & \\ x_r & x_r^3 & \dots & x_r^{2n-1} & \dots & x_r^{2n+2r-1} & 0 & 0 & 0 \dots & (-1)^{n+1} \\ A_1 & A_2 & \dots & A_n & \dots & A_{n+r} & 0 & 0 & 0 \dots & 0 \\ A_2 & A_3 & \dots & A_{n+1} & \dots & A_{n+r+1} & 0 & 0 & 0 \dots & 0 \\ & \dots & & \dots & & \dots & & \dots & \dots & \dots \\ A_n & A_{n+1} & \dots & A_{2n-1} & \dots & A_{2n+r-1} & 0 & 0 & 0 \dots & 0 \end{Bmatrix}$$

$$B \equiv \begin{Bmatrix} {}_{2n+2r-2}\alpha_2 & -{}_{2n+2r-2}\alpha_4 & \dots & (-1)^{n+r-1} & q_{2n+2r-1}(x_1) & \dots \\ {}_{2n+2r-4}\alpha_2 & -{}_{2n+2r-2}\alpha_4 & \dots & & -q_{2n+2r-3}(x_1) & \dots \\ \dots & \dots & \dots & & \dots & \dots \\ {}_{2n}\alpha_2 & -{}_{2n}\alpha_4 & \dots & & (-1)^{r-1}\, q_{2n+1}(x_1) & \dots \end{Bmatrix}$$

A and B have $n+2r$ columns each, and $n+r$ and r rows respectively. Here again the value of the determinant Δ of r rows and columns obtained by taking r consecutive columns from the end in B leaving aside the very last r columns containing the q's is $(-1)^{r(n+r-1)}$. Similarly the value of the determinant D obtained from A by taking the last r and the first n columns in A is $(-1)^{\frac{r(r+1)}{2}}$ $[A_1, A_n]$. Hence we easily derive the equation

$$\begin{vmatrix} x_1 & x_1^3 & \dots & x_1^{2n+2r-1} \\ x_2 & x_2^3 & \dots & x_2^{2n+2r-1} \\ \dots & \dots & & \dots \\ x_r & x_r^3 & \dots & x_r^{2n+2r-1} \\ A_1 & A_2 & \dots & A_{n+r} \\ A_2 & A_3 & \dots & A_{n+r+1} \\ \dots & \dots & & \dots \\ A_n & A_{n+1} & \dots & A_{2n+r-1} \end{vmatrix}$$

$$= (-1)^{nr}\ [A_1, A_n] \begin{vmatrix} q_{2n+1}(x_1) & q_{2n+1}(x_2) & \dots & q_{2n+1}(x_r) \\ q_{2n+3}(x_1) & q_{2n+3}(x_2) & \dots & q_{2n+3}(x_r) \\ \dots & \dots & \dots & \dots \\ q_{2n+2r-1}(x_1) & q_{2n+2r-1}(x_2) & \dots & q_{2n+2r-1}(x_r) \end{vmatrix} \quad \dots (21)$$

In exactly the same way, we find that

$$\begin{vmatrix} 1 & x_1^2 & x_1^4 & \cdots & x_1^{2n+2r-2} \\ 1 & x_2^2 & x_2^4 & \cdots & x_2^{2n+2r-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & x_r^2 & x_r^4 & \cdots & x_r^{2n+2r-2} \\ A_0 & A_1 & A_2 & \cdots & A_{n+r-1} \\ A_1 & A_2 & A_3 & \cdots & A_{n+r} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ A_{n-1} & A_n & A_{n+1} & \cdots & A_{2n+r-2} \end{vmatrix}$$

$$= (-1)^{nr} [A_0, A_{n-1}] \begin{vmatrix} q_{2n}(x_1) & q_{2n}(x_2) & \cdots & q_{2n}(x_r) \\ q_{2n+2}(x_1) & q_{2n+2}(x_2) & \cdots & q_{2n+2}(x_r) \\ \cdots & \cdots & \cdots & \cdots \\ q_{2n+2r-2}(x_1) & q_{2n+2r-2}(x_2) & \cdots & q_{2n+2r-2}(x_r) \end{vmatrix} \quad \ldots (22)$$

If $q_{n+1}(x_{r,s}) = x_{r,s} - {}_n\alpha_n\, x_{r,s-1}$

$$+ {}_n\alpha_{n-2}\, x_{r,s-2} - {}_n\alpha_{n-1}\, x_{r,s-3} + \ldots\ldots$$

so that $q_{2n}(x_{r,n+1}) = x_{r,n+1} - {}_{2n-1}\alpha_{2n-1}\, x_{r,n}$

$$+ {}_{2n-1}\alpha_{2n-3}\, x_{r,n-1} - \ldots + (-1)^n\, {}_{2n-1}\alpha_1 \cdot x_{r,1}.$$

$$q_{2n+1}(x_{r,n+1}) = x_{r,n+1} - {}_{2n}\alpha_{2n}\, x_{r,n}$$

$$+ {}_{2n}\alpha_{2n-2}\, x_{r,n-1} - \ldots + (-1)^n\, {}_{2n}\alpha_2 \cdot x_{r,1}.$$

then the following equations hold;

$$\begin{vmatrix} x_{11} & x_{12} & \cdots & x_{1,m+n} \\ x_{21} & x_{22} & \cdots & x_{2,m+n} \\ \cdots & \cdots & \cdots & \cdots \\ x_{m1} & x_{m2} & \cdots & x_{m,m+n} \\ A_0 & A_1 & \cdots & A_{m+n-1} \\ A_1 & A_2 & \cdots & A_{m+n} \\ \cdots & \cdots & \cdots & \cdots \\ A_{n-1} & A_n & \cdots & A_{m+2n-2} \end{vmatrix} = (-1)^{mn} [A_0, A_{n-1}] \,.$$

$$\times \begin{vmatrix} q_{2n}(x_{1,n+1}) & q_{2n}(x_{2,n+1}) & \cdots & q_{2n}(x_{m,n+1}) \\ q_{2n+2}(x_{1,n+2}) & q_{2n+2}(x_{2,n+2}) & \cdots & q_{2n+2}(x_{m,n+2}) \\ \cdots & \cdots & \cdots & \cdots \\ q_{2m+2n-2}(x_{1,m+n}) & q_{2m+2n-2}(x_{2,m+n}) & \cdots & q_{2m+2n-2}(x_{m,m+n}) \end{vmatrix} \quad \ldots (23)$$

$$\begin{vmatrix} x_{11} & x_{12} & \cdots & x_{1,m+n} \\ x_{21} & x_{22} & \cdots & x_{2,m+n} \\ \cdots & \cdots & \cdots & \cdots \\ x_{m1} & x_{m2} & \cdots & x_{m,m+n} \\ A_1 & A_2 & \cdots & A_{m+n} \\ A_2 & A_3 & \cdots & A_{m+n+1} \\ \cdots & \cdots & \cdots & \cdots \\ A_n & A_{n+1} & \cdots & A_{m+2n-1} \end{vmatrix} = (-1)^{mn}\,[A_1, A_n]$$

$$\times \begin{vmatrix} q_{2n+1}(x_{1,n+1}) & q_{2n+1}(x_{2,n+1}) & \cdots & q_{2n+1}(x_{m,n+1}) \\ q_{2n+3}(x_{1,n+2}) & q_{2n+3}(x_{2,n+2}) & \cdots & q_{2n+1}(x_{m,n+2}) \\ \cdots & \cdots & \cdots & \cdots \\ q_{2m+2n-1}(x_{1,m+n}) & q_{2m+2n-1}(x_{2,m+n}) & \cdots & q_{2m+2n-1}(x_{m,m+n}) \end{vmatrix} . \quad (24)$$

Equations (23) and (24) are the most general of the results so far obtained in this paper; and it is obvious that many of the earlier results are particular cases of these. There is another form of the value of the determinants in (23) and (24) which is advantageous in many ways. It evaluates $[A_{n-r}, A_n]$ in a form in which the actual calculation in any particular case is much simpler. We proceed to indicate briefly the lines of study. The matrices have to be arranged in a different way.

§ **7.** From the definition of $q_{n+1}(x_{r,s})$, we have

$$q_{2n}(x_{1,n+r}) = x_{1,n+r} - {}_{2n-1}\alpha_{2n-1}\, x_{1,n+r-1} + \cdots + (-1)^n\, {}_{2n-1}\alpha_1\, x_{1,r}$$

$$q_{2n+1}(x_{1,n+r}) = x_{1,n+r} - {}_{2n}\alpha_{2n}\, x_{1,n+r-1} + \cdots + (-1)^n\, {}_{2n}\alpha_2\, x_{1,r}$$

Consider the following A and B matrices which are easily seen to satisfy the conditions of our theorems.

$$A \equiv \left\{ \begin{array}{llllllll} x_{1,1} & x_{1,2} & \cdots & \cdots & x_{1,n} & x_{1,n+1} & \cdots & \\ \quad x_{1,m+n} & (-1)^{n+1} & 0 & & 0 & & \cdots & 0 \\ x_{21} & x_{22} & \cdots & \cdots & x_{2n} & x_{2,n+1} & \cdots & \\ \quad x_{2,m+n} & 0 & (-1)^{n+2} & & 0 & & \cdots & 0 \\ \cdots & & \cdots & & \cdots & & \cdots & \\ x_{m,1} & x_{m,2} & \cdots & \cdots & x_{m,n} & x_{m,n+1} & \cdots & \\ \quad x_{m,m+n} & & 0 & & 0 & & \cdots & (-1)^{n+m} \\ A_0 & A_1 & \cdots & \cdots & A_{n-1} & A_n & \cdots & \\ \quad A_{n+m-1} & 0 & 0 & & 0 & & \cdots & 0 \\ A_1 & A_2 & \cdots & \cdots & A_n & A_{n+1} & \cdots & \\ \quad A_{n+m} & 0 & 0 & & 0 & & \cdots & 0 \\ \cdots & & \cdots & & \cdots & & \cdots & \\ A_{n-1} & A_n & \cdots & \cdots & A_{2n-2} & A_{2n-1} & \cdots & \\ \quad A_{2n+m-2} & 0 & 0 & & 0 & & \cdots & 0 \end{array} \right\}$$

$$B \equiv \left\{ \begin{matrix} {}_{2n-1}\alpha_1 & -{}_{2n-1}\alpha_3 & \ldots & (-1)^{n-1}\,{}_{2n-1}\alpha_{2n-1} & (-1)^n & & 0\ldots \\ & & 0 & q_{2n}(x_{1,n+1}) & \ldots & (-1)^{m-1} q_{2n}(x_{m,n+1}) & \\ 0 & -{}_{2n}\alpha_2 & \ldots & (-1)^{n-1}\,{}_{2n}\alpha_{2n-2} & (-1)^n\,{}_{2n}\alpha_{2n} & (-1)^{n+1} & 0\ldots \\ & & 0 & -q_{2n+1}(x_{1,n+2}) & \ldots & (-1)^m q_{2n+1}(x_{m,n+2}) & \\ \ldots & & \ldots & & \ldots & \ldots & \end{matrix} \right\}$$

Hence

$$\begin{vmatrix} x_{11} & x_{12} & \ldots & x_{1m+n} \\ \ldots & \ldots & & \ldots \\ x_{m1} & x_{m2} & \ldots & x_{m,m+n} \\ A_0 & A_1 & \ldots & A_{m+n-1} \\ A_1 & A_2 & \ldots & A_{m+n} \\ \ldots & \ldots & & \ldots \\ A_{n-1} & A_n & \ldots & A_{m+2n-2} \end{vmatrix} = (-1)^{mn}[A_0 \; A_{n-1}].T$$

where

$$T \equiv \begin{vmatrix} q_{2n}(x_{1,n+1}) & q_{2n}(x_{2,n+1}) & \ldots & q_{2n}(x_{m,n+1}) \\ q_{2n+1}(x_{1,n+2}) & q_{2n+1}(x_{2,n+2}) & \ldots & q_{2n+1}(x_{m,n+2}) \\ q_{2n+2}(x_{1,n+3}) & q_{2n+2}(x_{2,n+3}) & \ldots & q_{2n+2}(x_{m,n+3}) \\ \ldots & \ldots & \ldots & \ldots \\ q_{2n+m-1}(x_{1,m+n}) & q_{2n+m-1}(x_{2,m+n}) & \ldots & q_{2n+m-1}(x_{m,m+n}) \end{vmatrix} \quad \ldots \quad (25)$$

Similarly

$$\begin{vmatrix} x_{11} & x_{12} & \ldots & x_{1m+n} \\ \ldots & & \ldots & \ldots \\ x_{m1} & x_{m2} & \ldots & x_{m,m+n} \\ A_1 & A_2 & \ldots & A_{m+n} \\ \ldots & & \ldots & \ldots \\ A_n & A_{n+1} & \ldots & A_{m+2n-1} \end{vmatrix} = (-1)^{mn}[A_1, A_n].T'$$

where

$$T' \equiv \begin{vmatrix} q_{2n+1}(x_{1,n+1}) & q_{2n+1}(x_{2,n+1}) & \ldots & q_{2n+1}(x_{m,n+1}) \\ q_{2n+2}(x_{1,n+2}) & q_{2n+2}(x_{2,n+2}) & \ldots & q_{2n+2}(x_{m,n+2}) \\ \ldots & \ldots & \ldots & \ldots \\ q_{2n+m}(x_{1,m+n}) & q_{2n+m}(x_{2,m+n}) & \ldots & q_{2n+m}(x_{m,+mn}) \end{vmatrix} \quad \ldots \quad (26)$$

The Indian Mathematical Society.

PROGRESS REPORT.

The following gentlemen have been enrolled as Honorary Members of the Society :—

(1) Prof. H. F. Baker of Cambridge.

(2) Prof. E. T. Whittaker of Edinburgh.

(3) Prof. A. N. Whitehead of London.

2. The following gentlemen have been admitted as Ordinary Members of the Society :—

(1) P. P. Anantanarayana Aiyar, M.A., Professor of Mathematics, Judson College, Rangoon.

(2) Balder Puri, M.A. (Cantab.) of Punjab.

3. The Calcutta Mathematical Society appears to be somewhat indifferent as regards holding a Mathematical Conference at Calcutta and the Managing Committee of our Society has decided to hold the Fourth Mathematical Conference at Poona in March next. Details of the arrangements will be announced in the next Progress Report. Meanwhile the members are hereby requested to prepare papers to be read before the Conference and send them with abstracts to the undersigned before the 31st January 1924.

P. V. SESHU AIYAR,
Hon. Secretary.

31*st August* 1923.

NOTICE.

The Librarian reports that the undermentioned periodicals issued in circulation have not yet been returned. Members who may happen to have the same are requested to kindly return them to the Librarian as early as possible.

P. V. SESHU AYYAR,
Hon. Secretary.

A. Current Journals.

(*N.B.*:—Numbers of the issues missing are written above the numbers of the Volumes to which they belong.)

(1) Messenger of Mathematics :- $\frac{11,\ 12}{41}, \frac{\text{All}}{42}, \frac{\text{All}}{43}, \frac{\text{All}}{44}, \frac{9,\ 10}{48}$.

(2) Quarterly Journal of Mathematics :- $\frac{3}{43}, \frac{4}{44}, \frac{\text{All}}{45}, \frac{1}{46}, \frac{1}{47}$.

(3) Mathematical Gazette :- $\frac{44,\ 48 \text{ to } 60}{3}, \frac{62,\ 63}{4}$.

(4) Annals of Mathematics :- $\frac{3}{3}, \frac{4}{14}, \frac{\text{All}}{15}, \frac{\text{All}}{16}, \frac{1}{17}, \frac{1.\ 2.\ 4}{18}, \frac{4,}{19}$

(5) American Journal of Mathematics :- $\frac{3}{33}, \frac{\text{All}}{37}, \frac{1}{38}, \frac{1,\ 3}{42}, \frac{4}{43}$.

(6) Bulletin of the American Mathematical Society :- $\frac{8,\ 9,\ 10}{15}, \frac{8}{23}, \frac{9}{26}$.

(7) Transactions of the American Mathematical Society :- $\frac{\text{All}}{15}, \frac{1,\ 2,\ 4}{16}, \frac{1}{17}, \frac{2}{18}, \frac{1}{20}, \frac{3,\ 4}{22}$.

(8) Monthly Notices of the Royal Astronomical Society :- $\frac{1,\ 2,\ 3,\ 4}{67}, \frac{2 \text{ to } 8}{72}, \frac{\text{All}}{73}, \frac{1 \text{ to } 4}{74}, \frac{1 \text{ to } 5,\ 7}{75}, \frac{\text{All}}{76}, \frac{\text{All}}{77}, \frac{4 \text{ to } 9}{78}, \frac{3}{79}, \frac{3 \text{ to } 6}{80}, \frac{9}{81}, \frac{1}{82}$.

(9) Proceedings of the Royal Society :- $\frac{594,\ 595}{87}, \frac{652}{93}, \frac{661}{94}, \frac{676}{96}, \frac{687}{97}$.

(10) Philosophical Magazine:- $\frac{127,\ 128}{22}, \frac{156}{26}, \frac{157}{27}, \frac{175}{30}, \frac{191,\ 192,}{32}, \frac{211}{36}, \frac{217}{37}, \frac{230,\ 231,\ 232}{39}, \frac{244}{41}, \frac{251,\ 252}{42}$.

(11) Astrophysical Journal :- $\frac{2 \text{ to } 5}{27}$, $\frac{\text{All}}{28}$, $\frac{3}{31}$, $\frac{1}{32}$, $\frac{3, 5}{35}$, $\frac{2}{37}$, $\frac{3 \text{ to } 5}{38}$, $\frac{2}{43}$, $\frac{5}{48}$, $\frac{1, 4, 5}{49}$, $\frac{2 \text{ to } 4}{50}$, $\frac{2}{53}$, $\frac{4}{54}$.

(12) Crele's Journal :- $\frac{2}{135}$, $\frac{3}{142}$, $\frac{\text{All}}{152}$.

(13) L' Intermediaire Mathematiciens:- $\frac{10}{15}$, $\frac{11, 12}{21}$, $\frac{\text{All}}{22}$, $\frac{\text{All}}{23}$, $\frac{\text{All}}{24}$, $\frac{1 \text{ to } 4}{25}$, $\frac{9 \text{ to } 12}{27}$, $\frac{\text{All}}{28}$.

(14) Mathematische Annalen :- $\frac{\text{All}}{76 \text{ to } 81}$.

(15) Philosophical Transactions of the Royal Society of London :— $\frac{505 \text{ to } 507}{213}$, $\frac{509, 510, 515, 516}{214}$, $\frac{541}{216}$, $\frac{549, 551}{217}$, $\frac{562, 563, 566}{218}$.

(16) Acta Mathematica : - $\frac{\text{All}}{39}$, $\frac{3, 4}{43}$.

(17) Popular Astronomy :- $\frac{3, 4, 5}{27}$, $\frac{3}{28}$, $\frac{3, 9}{29}$.

(18) Proceedings of the Edinburgh Mathematical Society :— $\frac{\text{All}}{32}$, $\frac{1}{35}$, $\frac{1}{36}$, $\frac{\text{All}}{37}$, $\frac{\text{All}}{38}$.

(19) Proceedings of the London Mathematical Society :— Series 1: $\frac{2, 7}{15}$, $\frac{4, 6, 7}{16}$; Series 2: $\frac{\text{All}}{1, 2, 4, 8, 9}$, $\frac{1 \text{ to } 5}{11}$, $\frac{2, 5, 6, 7}{13}$, $\frac{1 \text{ to } 4}{14}$, $\frac{6}{19}$.

(20) Mathematics Teacher :- $\frac{\text{All}}{6}$, $\frac{2}{11}$, $\frac{2}{12}$, $\frac{2}{14}$.

(21) Bulletin of the Calcutta Mathematical Society :- $\frac{\text{All}}{2, 4, 6}$, $\frac{3}{11}$, $\frac{2}{12}$.

(22) The Tohoku Mathematical Journal :- $\frac{4}{11}$, $\frac{3, 4}{14}$.

(23) Nature :— $\frac{2318, 2319, 2327}{93}$, $\frac{2341, 2349, 2350, 2351}{94}$, $\frac{2374, 2376, 2384 \text{ to } 2387, 2391}{95}$, $\frac{2392 \text{ to } 2394, 2396, 2398, 2417}{96}$, $\frac{2458, 2460}{98}$, $\frac{2482, 2491}{99}$

$\frac{2493,\ 2494,\ 2496,\ 2505,\ 2508,\ 2509,\ 2512 \text{ to } 2515}{100}$, $\frac{2538}{101}$, $\frac{2578,\ 2583,\ 2589}{103}$, $\frac{2622,\ 2626}{104}$, $\frac{2627,\ 2637,\ 2651,\ 2652}{105}$, $\frac{2713,\ 2714,\ 2717}{108}$, $\frac{2742,\ 2743}{109}$.

(24) The American Mathematical Monthly :— $\frac{4}{24}$, $\frac{1}{25}$.

(25) Proceedings of the Benares Mathematical Society :—Only Part II of Vol. II of 1920 is received in the Library on the 2nd of September 1921.

(26) School Science and Mathematics :—$\frac{1 \text{ to } 5,\ 7 \text{ to } 9}{7}$, $\frac{\text{All}}{9}$, $\frac{8,\ 9}{10}$, $\frac{3}{17}$, $\frac{6,\ 9}{18}$, $\frac{\text{All}}{19}$, $\frac{1,\ 2}{20}$, $\frac{4,\ 9}{21}$.

B. Discontinued Journals.

(*N.B.*:—The numbers of the issues missing are written above the numbers of the volumes to which they belong.)

(1) Educational Times :- $\frac{664}{70}$.

(2) Transactions of the Cambridge Philosophical Society :- $\frac{6,\ 7,\ 9}{22}$.

(3) Fortschritte de Mathematik :- $\frac{1,\ 2}{35}$, $\frac{3}{36}$.

(4) Bulletin des Sciences Mathematiques :- $\frac{12}{33}$, $\frac{12}{39}$.

(5) Mathesis :- $\frac{6,\ 7,\ 8}{8 \text{ of } 1908}$, $\frac{9,\ 10,\ 12}{1 \text{ of } 1911}$, $\frac{4 \text{ to } 12}{2 \text{ of } 1912}$, $\frac{1,\ 2}{1913}$.

(6) Journal des Mathematiques Elementaires :— $\frac{17,\ 18}{35}$, $\frac{1 \text{ to } 7,\ 10 \text{ to } 16}{36}$, $\frac{\text{All}}{37}$, $\frac{1,\ 3 \text{ to } 12}{38}$.

(7) Comptes Rendus :- $\frac{\text{All}}{145}$.

(8) L'Education Mathematique :— $\frac{16}{12}$, $\frac{\text{All}}{13}$, $\frac{18,\ 19,\ 20}{14}$, $\frac{\text{All}}{15}$, $\frac{13,\ 14}{16}$.

(9) Journal de Mathematique Pure et Appliques :- $\frac{2}{5 \text{ of } 1909}$, $\frac{\text{All}}{10 \text{ of } 1914}$.

THE INDIAN MATHEMATICAL SOCIETY.

Statement of Accounts for the Year 1922.

Receipts :—	Rs.	A.	P.	Expenditure :—	Rs.	A.	P.
Balance from 1921	7,978	4	5	Ordinary working expenses... ...	323	12	...
Subscription from Members... ...	679	14	...	Books and periodicals	70	11	...
Life Member's subscription	150	...	...	Printing Journal	823	14	...
Subscriptions for Journal	187	15	...	Library expenses	389	4	3
Interest on investments	336	14	5	Closing balance	7,752	6	7
Miscellaneous	27	...	...				
	9,359	15	10		9,359	15	10

MADRAS,

October 1923.

S. NARAYANA AIYAR,

Hon. Treasurer.

Equations (25) and (26) are simpler than (23) and (24), since in (24) the highest q is $q_{2n+2m-1}$, whereas in (26), it is q_{2n+m}. The corresponding equations for the results contained in (22), (21) and others may be easily written down. We content ourselves with obtaining the equations corresponding to (13) and (14). Thus in (25) and (26) equate the coefficients of x_{11}, x_{22}, ... x_{mm}; we obtain from both equations

$$[A_m, A_{m+n-1}] = [A_o, A_{n-1}] \times \begin{vmatrix} {}_{2n-1}a_1 & {}_{2n-1}a_3 & & \cdots & {}_{2n-1}a_{2m-1} \\ 0 & {}_{2n}a_2 & {}_{2n}a_4 & \cdots & {}_{2n}a_{2m-2} \\ 0 & {}_{2n+1}a_1 & {}_{2n+1}a_3 & \cdots & {}_{2n+1}a_{2m-3} \\ 0 & 0 & {}_{2n+2}a_2 & \cdots & {}_{2n+2}a_{2m-4} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & & & {}_{2n+m-2}a_m . \end{vmatrix} \quad \ldots \quad (27)$$

If we remember that $[A_0, A_{n-1}]\ {}_{2n-1}a_1 = [A_1, A_n]$, the determinant on the right having m rows and columns

$$= [A_1, A_n] \begin{vmatrix} {}_{2n}a_2 & {}_{2n}a_4 & \cdots & {}_{2n}a_{2m-2} \\ {}_{2n+1}a_1 & {}_{2n+1}a_3 & \cdots & {}_{2n+1}a_{2m-3} \\ 0 & {}_{2n+2}a_2 & \cdots & {}_{2n+2}a_{2m-4} \\ 0 & {}_{2n+3}a_1 & \cdots & {}_{2n+3}a_{2m-5} \\ 0 & 0 & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots \\ \cdots & \cdots & & \cdots \end{vmatrix} \cdots \quad \ldots \quad (28)$$

of $(n-1)$ rows and columns.

Ex. $[E_3, E_5] = [E_o, E_2] \cdot \begin{vmatrix} {}_5a_1 & {}_5a_3 & {}_5a_5 \\ 0 & {}_6a_2 & {}_6a_4 \\ 0 & {}_7a_1 & {}_7a_3 \end{vmatrix}$

$$= (4!)^2.(2!)^2.225. \begin{vmatrix} {}_6a_2 & {}_6a_4 \\ {}_7a_1 & {}_7a_3 \end{vmatrix}$$

$$= 2^2.24^2.225. \begin{vmatrix} 3429 & 1519 \\ 11025 & 24940 \end{vmatrix}$$

$$= 2^2.24^2.225.68,772,285.$$

APPENDIX.

Table for 'α' numbers.

N.B.— ${}_{n+2r}\alpha_n$ is the nth number in the $(r+1)$th even column.

a_1					
a_2					
a_3	Σa_1	$a_3\Sigma a_1$			
a_4	Σa_2	$a_4\Sigma a_2$			
a_5	Σa_3	$a_5\Sigma a_3$	$\Sigma a_3\Sigma a_1$	$a_5\Sigma a_3\Sigma a_1$	
a_6	Σa_4	$a_6\Sigma a_4$	$\Sigma a_4\Sigma a_2$	$a_6\Sigma a_4\Sigma a_2$	
a_7	Σa_5	$a_7\Sigma a_5$	$\Sigma a_5\Sigma a_3$	$a_7\Sigma a_5\Sigma a_3$	$\Sigma a_5\Sigma a_3\Sigma a_1$
a_8	Σa_6	$a_8\Sigma a_6$	$\Sigma a_6\Sigma a_4$	$a_8\Sigma a_6\Sigma a_4$	$\Sigma a_6\Sigma a_4\Sigma a_2$

Same Table when $\alpha_n = n^2$.

1							
4							
9	1	9					
16	5	80					
25	14	350	9	225			
36	30	1030	89	3204			
49	55	2695	439	21511	225	11025	
64	91	5824	1519	97216	3429	219456	
	140		4214		24940		11025
	204		10038		122156		230481

Easy Group Theory.

Professor G. A. Miller, of the University of Illinois, has contributed to the *Scientific Monthly** a popular exposition of the 'Theory of Groups' in an article entitled *Easy Group Theory.* The fact that the University of Paris recently founded a Chair of "Theory of Groups and Calculus of Variations" has given occasion to the writer to draw the attention of scientific men to the 'essence of a mathematical group and the rôle which the group concept is assuming in modern mathematical developments.'

It is often remarked that the 'Group Theory' is a kind of 'mathematical luxury,' seeing that the scientific world got on without it for centuries. But this can be said of all new developments, to a greater or less degree. At any rate, it can be safely asserted that a knowledge of the group theory greatly adds to the appreciation of beauty in mathematical work and gives a deeper insight into the conditions of an intellectual problem and a better grasp of its solution.

The following extracts are intended to convey to the reader the purport of the article in question :—

"Paris may justly claim a very large share in the early development of group theory. A Parisian, A. L. Cauchy, is commonly regarded as the founder of this subject, while other Parisians, including A. T. Vandermonde and E. Galois, did important pioneering work in this field. Although J. L. Lagrange is commonly regarded as a French mathematician, his pioneering work in group theory was done before he settled in Paris. The first separate treatise on this subject was written by a Parisian. This work appeared in 1870 under the title *Traité des substitutions et des équations algébriques* by C. Jordan.

In group theory the first extensive formal or abstract developments are due to English and German mathematicians. Notwithstanding the fundamental importance of formal developments and abstract formulations, the real life of mathematics abides in its contact with the concrete. In the case of group theory this contact has been emphasized especially by S. Lie, F. Klein and H. Poincare. The rapidly increasing promi-

* Vol. XV, No. 6, Dec. 1922.

nence of this theory during the last half century is largely due to the writings of these three men, who have been leaders also in various other lines of mathematical activity.

A dominating mathematical concept, like a dominating personality, has a charm of its own, and creates a kind of atmosphere which is as invigorating as that of a real university or that which emanates from any group of real scholars. A fundamental notion of a mathematical group is that " there is no new thing under the sun." It is true that we speak here of generators and generational relations, but the objects which are generated were members of the group since the beginning of time and will remain members thereof throughout eternity. We study the group to perceive this sameness in its various forms and to understand the relative properties of the various elements which unite to make a single element of the group.

The non-technical meaning of the term 'group' suggests little as regards its technical meaning, except the invariance of the number of its elements. In a non-technical group, one usually thinks of the elements as units which may or may not have the power of reproduction. These elements are usually new elements of the group. In the technical group, the elements have necessarily the power of uniting; but when they unite they neither produce any new element nor lose their own identity. The union merely exhibits the possible production of an element of the group, or the different ways of securing *e pluribus unum* Union, unity and stability constitute the triumvirate of the theory of groups. The stability here noted is not the stability of statics but the stability of dynamics. It is a kind of invariance under transformations.

What is perhaps of most interest in this connection as regards the non-mathematician is the question why the concept to which we refer above is so fundamental in mathematics. It is well known that mathematical developments have been largely guided by the desire to secure intellectual penetration into the workings of nature. Do we find in nature numerous instances of the union of elements of the same kind to produce an element of this kind which is really not new but belongs to a totality which has been clearly defined? For instance, one may think of the totality of the transformations of space subject to the condition that the distance between every pair of points in space remains invariant. It is evident that if one such transformation is followed by another the two together are equivalent to a single transformation of the totality in question. Hence we say that this totality constitutes a group.

It is also clear that the totality of the rational integral numbers when they are combined by addition has the property that no new number arises from the combination of any two of them, or from the combination of one with itself. In this totality and in the totality of transformations noted in the preceding paragraph, it is evident that if x, y, z represent three elements of the totality and if any two of them are supposed to be known then the third is completely determined by the following equation :

$$x\ y = z.$$

The term group is commonly used in mathematics in the restricted sense that the third of any three elements is completely determined by such an equation where any two of them are supposed to be given. Moreover, it is usually assumed that when any three elements are combined the associative law is satisfied, but it is not assumed that the commutative law is necessarily satisfied when two of them are combined.

Even with these restrictions there are instances of groups almost wherever one turns. In the vegetable and the animal kingdoms one sees new elements arising in profusion, but if one looks deep enough these are cases of a union of elements without loss of identity of the elements, and there seems to be nothing new in the profound formal physical sense. Hence one may see some significance in the following statement made by Poincaré shortly before his death : "The theory of groups is, so to say, entire mathematics, divested of its matter and reduced to a pure form."

In the groups noted above the number of elements is infinite. As instances of a finite group we may consider the six movements which transform a fixed equilateral triangle into itself and the eight movements which transform a fixed square into itself. Such special instances are, of course, of little interest to the general scientist except as illustrations. The thing that may be supposed to command the interest even of the educated layman is the fact that these very evident considerations appertain to a profound mathematical theory.

Group theory did not arise from such obvious considerations. After it was partially developed as an autonomous science, some of its more obvious applications received special attention. A certain degree of difficulty always seems to create the most favourable atmosphere for scientific developments. In group theory this difficulty was created by the n roots of the algebraic equation of the following form :

$$x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_n = 0.$$

It is customary to speak of x as the unknown and to call this an equa-

tion of degree n in one unknown. As a matter of fact the equation has n roots and all of these are unknown, so that it is really an equation in n unknowns. The constant co-efficients $a_1, a_2 \ldots a_n$ were known to be symmetric functions of these n unknowns before the subject of group theory was developed.

The mysterious appearance of n unknowns to take the place of the one which presents itself openly is perhaps sufficient to create an atmosphere suitable for scientific endeavours. At any rate, it was in this atmosphere that our subject arose and hence we shall note here a few of the early steps in its development. It is true that these steps were taken by men who seemed to have no idea that they were dealing with notions which had the widest applications in other fields of mathematical investigation. In fact, the early explorers of group theory died long before it was realized that the notions with which they were dealing were destined to permeate a large part of the science of mathematics.

The n roots of an equation of degree n with constant co-efficients constitute a group in the non-technical sense of the term; but the group of this equation is something very different and lies much deeper. It relates to a certain totality of permutations of these n roots, or substitutions on these n roots, leaving invariant every possible rational function of these roots which is equal to a constant, and having the property that if a rational function of these roots is invariant under these substitutions it is equal to a constant. These fundamental properties of the n roots of the equation in question were first noted by E. Galois, a French mathematician of great renown, although he died before reaching the age of twenty-one years. They were sufficiently difficult to create an atmosphere suitable for the development of our subject as an autonomous science.

A study of the permutations of n things might at first sight appear to promise little of importance. It is true that before the time of Galois attention had been called to this subject by Lagrange and Vandermonde in connection with the question: Under what condition a rational function of n variables can be expressed rationally in terms of another such function? but it was not until long after the days of Galois that mathematicians began to realize the fundamental importance of this subject in the study of a large variety of mathematical questions. When the substitutions arising from these various permutations were studied by themselves they were seen to combine according to laws which are found almost everywhere when the data are sufficiently connected to admit of mathematical treatment.

The reader who has given some attention to mathematical developments may be inclined to ask : If the notion 'group' is so fundamental, why did not the ancients and even such eminent later thinkers as Descartes and Newton pay explicit attention to it ? Why does not the theory of groups have an ancient prototype like differential and integral calculus, whose prototype is found in the method of exhaustion of the ancient Greeks ? Does the history of mathematics present any other instance of the sudden rise of a dominating concept extending into practically all the large branches of mathematics ?

The fact that the last of these questions must be answered in the negative tends to enhance the interest in the others. This negative answer calls also for a word of caution ; for, there is danger that the reader might infer from it that the subject of group theory has greater merit than really belongs to it. It seems that mathematical developments have always been guided by the group concept. In the words of Poincarè "the ancient mathematicians employed groups in many cases without knowing it." The main question in mathematical developments is that one gets on the right road. The ability to explain why one has chosen this road is of secondary importance. In fact, the best teacher is frequently unable to give a good account of his methods, while a much inferior teacher may be able to talk glibly about methods.

Group theory is largely a method and those who are studying this subject by itself may be compared to those who are devoting their attention to methods of teaching. Just as the latter are not necessarily the best teachers, so the former are not necessarily the best mathematical investigators. Possibly, the bacteria which have tended to make the teachers in colleges such a prominent feature of our modern universities have also caused the emphasis on group theory in modern mathematics. Just as some of our best teachers have never read a work on methods of teaching, so some of the best mathematical investigators have never secured a speaking acquaintance with the notion of group. In both cases the real essence of the subject has been acquired unconsciously, or, at least, without the development of a formal language relating thereto.

The fact that modern mathematicians emphasize the group concept while the ancient and mediæval mathematicians did not do this does not imply a change of mental attitude. Naturally the modern mathematicians secured a somewhat deeper insight into various subjects and thus discovered evidences of groups which were unknown to the older mathematicians. Inspired by these groups they developed also the

theory relating to the groups which the ancients used implicitly, but it is questionable whether the modern mathematicians would have developed a theory relating to the latter if they had not been inspired by the former. At any rate, they did not take any steps towards such a theory before they had this additional motive. These observations may serve as partial answers to questions raised above relating to the late development of our subject as an autonomous science.

The heading of the article suggests that some of the developments of our subject can not be properly called easy. In fact, by far the larger part of these developments presupposes a rather extensive technical knowledge and hence they are unsuited for a popular article. Among all the scientists, the mathematician works usually at the greatest distance from his postulates, and hence has the greatest difficulty to exhibit the results of his toil to the public so as to secure appreciation. In group theory this distance has become especially long even for a mathematical subject. The present article aims merely to direct attention to the richness of the mathematical developments which have contact with the experiences of all thoughtful persons and thus to secure an easier approach to some of this richness.

If a group contains a finite number of elements this number is called the *order* of the group. For instance, the 24 different movements of space which transform a cube as a whole into itself but interchange some of its parts constitute a group of order 24. The most elementary group of a given order g is cyclic ; that is, it is composed of the powers of a single one of its elements. For instance, the g numbers which satisfy the condition that the gth power of each of them is equal to unity constitute this group of order g, where g is any natural number. It is evident that the 24 different movements which transform a cube into itself are not powers of a single one of them. Hence they constitute a non-cyclic group of order 24. In fact, none of the elements of this group has to be raised to a higher than the fourth power in order to obtain unity, or the identity.

One of the fundamental problems of abstract group theory is the determination of all the possible groups of a given order g. It was noted above that there is one and only one cyclic group of every possible order g. When g is a prime number there is no other group of this order. This is also sometimes the case when g is composite but there is no upper limit to the number of groups which may have the same composite order. That is, it is always possible to find a number g such that the number of the different abstract groups of order g exceeds any given

Notes and Questions.

A Note in Analytical Geometry.

1. If the equation to a straight line be

$$lx + my + n = 0$$

we know that the equation of the diameter conjugate to the line with respect to the conic

$$f(x, y) \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

is
$$X/l = Y/m, \text{ or } mX - lY = 0,$$

where
$$X \equiv ax + hy + g, \quad Y \equiv hx + by + f.$$

In other words, the diameter conjugate to a given line is obtained by substituting $(-Y, X)$ for (x, y) in the original equation and suppressing the constant.

Hence given a set of straight lines

$$u_n + u_{n-1} + \dots\dots\dots + u_0 = 0;$$

the diameters conjugate to the set are written down at once by replacing (x, y) in u_n by $(-Y, X)$.

Applications:

(i) *Asymptotes*: The asymptotes of $f(x, y) = 0$ are of the form $f(x, y) = k$ and are self-conjugate diameters. Hence their equation is

$$a(-Y)^2 + 2hX(-Y) + bX^2 = 0;$$

i.e.
$$aY^2 - 2hXY + bX^2 = 0,$$

as is well known.

(ii) *Axes*: The axes of a conic being the bisectors of the angles between the asymptotes are, from the above,

$$\frac{X^2 - Y^2}{a - b} = \frac{XY}{h}.$$

They may also be deduced *directly* as under: Since the axes may be defined as the diameters conjugate to the straight lines bisecting the angles between the asymptotes, their equations are obtained by substituting $(-Y, X)$ for (x, y) in

$$\frac{x^2 - y^2}{a - b} = \frac{xy}{h}.$$

That is, the axes are

$$\frac{Y^2 - X^2}{a - b} = -\frac{YX}{h}$$

or

$$\frac{X^2 - Y^2}{a - b} = \frac{XY}{h}.$$

2. Similarly in *areals*, if $lx + my + nz = 0$ be any straight line, the diameter conjugate to it with respect to the conic

$$\phi(x, y, z) \equiv Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy = 0$$

is seen to be

$$l(Y - Z) + m(Z - X) + n(X - Y) = 0,$$

where X, Y, Z denote $\frac{1}{2}\frac{\partial\phi}{\partial x}, \frac{1}{2}\frac{\partial\phi}{\partial y}, \frac{1}{2}\frac{\partial\phi}{\partial z}$ respectively.

Hence, we have the following :

Given a set of straight lines in areals, the diameters conjugate to the set are obtained by replacing (x, y, z) in the equation of the lines by $(Y - Z, Z - X, X - Y)$.

Applications :—

(i) *Asymptotes*: The equation

$$l(Y - Z) + m(Z - X) + n(X - Y) = 0$$

is written

$$\Sigma x(m - n)(A + F - G - H) - (x + y + z)\Sigma A(m - n) = 0. \quad \text{(I)}$$

Identifying this with

$$lx + my + nz = 0,$$

and remembering that

$$(m - n) : (n - l) : (l - m) = (Y - Z) : (Z - X) : (X - Y),$$

we have

$$\frac{(m - n)(A + F - G - H) - \Sigma A(m - n)}{l} = \&c. = \&c.,$$

$$\therefore \frac{(m - n)(A + F - G - H) - (n - l)(B + G - H - F)}{l - m} = \&c. = \&c.$$

i. e. $$\frac{(Y - Z)(A + F - G - H) - (Z - X)(B + G - H - F)}{X - Y}$$

$$= \&c. = \&c :$$

whence we get the asymptotes in the symmetrical forms

$$\Sigma\,(A+F-G-H)\,(Y-Z)^2=0;$$

$$\Sigma\,(B+C-2F)\,(X-Y)\,(Z-X)=0.$$

[See: Clement Jones: *Alg. Geo.*, p. 478.]

(ii) *Axes*: From (I) above we see that the equation

$$\Sigma\, x(m-n)\,(A+F-G-H)=0 \qquad \ldots \quad \text{(II)}$$

represents a line parallel to the diameter conjugate to

$$lx+my+nz=0. \qquad \ldots \qquad \ldots \quad \text{(III)}$$

Hence to find the axes, we put down the condition that (II) and (III) may be perpendicular to each other, *viz.*:

$$\begin{vmatrix} (m-n)\cot A, & (n-l)\cot B, & (l-m)\cot C \\ (m-n)\,(A+F-G-H), & \&c., & \&c. \\ 1, & 1, & 1 \end{vmatrix}=0$$

Replacing

$$(m-n),\ (n-l),\ (l-m) \text{ by } (Y-Z),\ (Z-X),\ (X-Y)$$

we get the equation of the axes in the form

$$\begin{vmatrix} \cot A & \cot B & \cot C \\ (A+F-G-H) & \&c. & \&c. \\ \left(\dfrac{1}{Y-Z}\right) & \left(\dfrac{1}{Z-X}\right) & \left(\dfrac{1}{X-Y}\right) \end{vmatrix}=0.$$

[See: Clement Jones: *Alg. Geo.*, p. 505.]

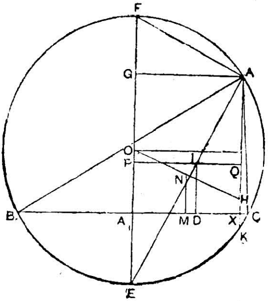
(iii) *General*: If $F(x, y, z)=0$ denotes the axes or the asymptotes of the conic $\phi(x,y,z)=0$, then $F(Y-Z, Z-X, X-Y)=0$ will also denote the same.

M. T. N.

195317

An Elementary Proof of Feuerbach's Theorem.

Let O be the centre of the circum-circle of the ΔABC, A_1 the mid-point of BC and EA_1 OF the diameter perpendicular to BC. Draw AX perpendicular to BC and in it take H, the ortho-centre of the Δ : produce AX to meet the circumcircle in K. Join EA which contains the incentre I; draw ID perpendicular to BC, and PIQ parallel to BC meeting EF, AX in P, Q. Join OH and bisect it in N, the nine-point centre; draw NM perpendicular to BC. Join AF and draw AG perpendicular to EF; also draw OY perpendicular to and bisecting AK.



(1) From the similar triangles AIQ, AFG

$$IQ : IA = FG : AF$$

Also, from the similar triangles EIP, EFA

$$IP : IE = AF : FE$$

$$\therefore \quad PI \,.\, IQ : AI \,.\, IE = FG : FE.$$

But $\quad AI \,.\, IE = 2Rr$

$$\therefore \quad PI \,.\, IQ = r \,.\, FG. \qquad \ldots \qquad \ldots \quad (1)$$

(2) In the isosceles Δ NPQ

$$NI^2 = NQ^2 - PI \,.\, IQ$$
$$= NQ^2 - r \,.\, FG, \text{ by (1)} \quad \ldots \qquad \ldots \quad (2)$$

(3) Now from the Δ NQX,

$$NQ^2 = NX^2 + QX^2 - 2QX \,.\, NM$$
$$= \tfrac{1}{4} R^2 + r^2 - 2r \,.\, NM$$

And $2.NM = OA_1 + HX = \tfrac{1}{2}(AH + HK) = AY$

$$\therefore \quad NQ^2 = \tfrac{1}{4} R^2 + r^2 - r \,.\, AY. \qquad \ldots \quad (3)$$

(4) Hence, combining (2) and (3),

$$NI^2 = \tfrac{1}{4} R^2 + r^2 - r\,(AY + FG)$$
$$= \tfrac{1}{4} R^2 + r^2 - rR$$
$$= (\tfrac{1}{2} R - r)^2. \quad \ldots \quad \ldots \quad \ldots \quad (4)$$
$$\therefore \quad NI = (\tfrac{1}{2} R - r),$$

which proves the theorem for the in-circle.

The proof for an ex-circle is quite similar to the above.

[I am indebted to Mr. Nandlal M. Mehta, B.A., for assistance in the writing of this Note].

K. J. SANJANA.

Maximum inscribed Parallelogram.

The following is a new proof of the theorem: "The maximum parallelogram inscribed in a triangle is the one formed by parallels to two of its sides through the middle point of the third side."

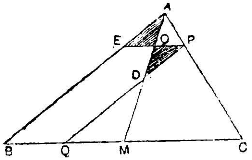
Let BEPQ be any parallelogram inscribed in the ΔABC.

Let the median AM cut EP at O and PQ at D.

Then $\Delta POD \equiv \Delta AOE$.

$\therefore$ Area of BEPQ = Area of BADQ

$= \tfrac{1}{2}\,\Delta\,ABC - \Delta QDM,$



which is a maximum when ΔQDM disappears; that is, when the parallels are drawn through the middle point of AC.

V. V. SATHYANARAYANA MURTHY

Note on Combinatory Analysis.

1. Starting with the well-known identity

$$\binom{n+1}{p+1} = \binom{n}{p} + \binom{n-1}{p} + \binom{n-2}{p} + \ldots +1, \quad \ldots (1)$$

where $\binom{m}{s}$ denotes the number of combinations of m things s at a time, we deduce the identity

$$\binom{n+1}{p+q+1} = \sum \left\{ \binom{n-k}{p} \cdot \binom{k}{q} \right\}, \; k \nless q, (n-k) \nless p. \quad \ldots (2)^*$$

To prove (2) by mathematical induction, we proceed thus :—

Assume it to be true for all values of n and q, and for a particular value of p. Then, by (1),

$$\binom{n+1}{p+q+2} = \binom{n}{p+q+1} + \binom{n-1}{p+q+1} + \ldots\ldots +1$$

$$= \sum \binom{n-k-1}{p} \cdot \binom{k}{q} + \sum \binom{n-k-2}{p} \cdot \binom{k}{q}$$

$$+ \ldots\ldots, \text{ by hypothesis.}$$

$$= \sum \left[\left\{ \binom{n-k-1}{p} + \binom{n-k-2}{p} + \ldots \right\} \cdot \binom{k}{q} \right]$$

$$= \sum \binom{n-k}{p+1} \cdot \binom{k}{q}, \text{ by (1).}$$

Hence, the identity (2) holds good for $(p+1)$, if it holds good for p. Obviously (2) is true when $p = 0$. Hence it is true when $p = 1$, and so on, universally.

2. Further extensions of (2) will follow by writing $(n+1)$ for n and $(q+r+1)$ for q. Thus we have

$$\binom{n+2}{p+q+r+2} = \sum \binom{n-k}{p} \cdot \binom{k+1}{q+r+1}, \text{ by (2).}$$

But $$\binom{k+1}{q+r+1} = \sum \binom{k-l}{q} \cdot \binom{l}{r}, \; [\, l \nless r, \; k-l \nless q.$$

$$\therefore \quad \binom{n+2}{p+q+r+2} = \sum \binom{n-k}{p} \cdot \binom{k-l}{q} \cdot \binom{l}{r}; \quad \ldots (3)$$

and so on.

---

* This result was suggested to the writer by Prof. E. B. Ross to whom the writer's acknowledgments are due.

Generally

$$\left\{ {n + m \atop \Sigma(p) + m} \right\} = \Sigma \left\{ \binom{n - k_1}{p_1} \binom{k_1 - k_2}{p_2} \binom{k_2 - k_3}{p_3} \cdots \right.$$

$$\left. \cdots\cdots \binom{k_{m+1}}{p_{m+1}} \right\} \quad \cdots (4)$$

Cor. 1. If all the p's are equal to unity, we can write

$$\binom{n + m}{2m + 1} = \Sigma (a_1 \; a_2 \cdots\cdots a_{m+1}), \quad \cdots \quad \cdots (5)$$

where $a_1 + a_2 + \cdots\cdots + a_{m+1} = n.$

In the theory of partitions this result may be stated thus:—

"*The sum of the products of all the partitions of a number N into* $(m + 1)$ *parts is the number of combinations of* $(N + m)$ *taken* $(2m + 1)$ *at a time.*"

I. TOTADRI IYENGAR.

[Results (1), (2), (3), can be *directly* obtained by equating the co-efficients (x^p), $(x^p y^q)$, $(x^p y^q z^r)$, in the identities:—

(i) $$\frac{(1 + x)^{n+1} - 1}{(1 + x) - 1} = (1 + x)^n + (1 + x)^{n-1} + \ldots + 1;$$

(ii) $$\frac{(1 + x)^{n+1} - (1 + y)^{n+1}}{(1 + x) - (1 + y)} = (1 + x)^n + (1 + x)^{n-1}(1 + y) + \cdots\cdots + (1 + y)^n;$$

(iii) $$\frac{(1+x)^{n+2}(y-z) + (1+y)^{n+2}(z-x) + (1+z)^{n+2}(x-y)}{(1+x)^2(y-z) + (1+y)^2(z-x) + (1+z)^2(x-y)} = \Sigma (1 + x)^{n-k} . (1 + y)^{k-l} . (1 + z)^l$$

—Ed.]

Solutions.

Question 1217.

(K. J. SANJANA):—(*a*) In a plane triangle ABC, if w is the Brocard angle, prove that

$$\sin^2 w \; \{ \cot^3 w + \cot (A - w) \cot (B - w) \cot (C - w) \}$$
$$= \operatorname{cosec} A \sin B \sin C + \operatorname{cosec} B \sin C \sin A + \operatorname{cosec} C \sin A \sin B.$$

(*b*) If G is the centroid, prove that

$$\cot GAB \cot GBC . \cot GCA + \cot GAC \cot GCB \cot GBA = 6 \cot w.$$

Additional Solution by M. V. Ramakrishnan.

(*a*) We know that

$$\sin^3 w = \sin (A - w) \sin (B - w) \sin (C - w).$$

$$\therefore \quad \cot^3 w + \cot (A - w) \cot (B - w) \cot (C - w)$$
$$= \operatorname{cosec}^3 w \; \{ \cos^3 w + \cos (A - w) \cos (B - w) \cos (C - w) \}$$
$$= \operatorname{cosec}^3 w \; \{ 3 \cos w \sin^2 w + \cos 3w + \cos (A+B+C-3w) + \Sigma \cos (A - w) \sin (B - w) \sin (C - w) \}$$
$$= 3 \cot w + \Sigma \cot (A - w)$$
$$= \{ \cot w + \cot (A - w) \} + \{ \cot w + \cot (B - w) \} + \{ \cot w + \cot (C - w) \}$$
$$= \Sigma \frac{\operatorname{cosec}^2 w}{\cot B + \cot C}$$
$$= \operatorname{cosec}^2 w \; \Sigma \frac{\sin B \sin C}{\sin A}.$$

Hence $\quad \sin^2 w \; \{ \cot^3 w + \cot (A - w) \cot (B - w) \cot (C - w) \}$
$$= \Sigma \operatorname{cosec} A \sin B \sin C.$$

(*b*) $\cot GAB \cot GBC \cot GCA + \cot GAC \cot GCB \cot GBA$
$$= \Sigma \cot GAB + \Sigma \cot GBA,$$

since $\quad GAB + GBC + GCA = \pi - GAC - GCB - GBA,$

and $\quad \sin GAB \sin GBC \sin GCA = \sin GAC \sin GCB \sin GBA.$

Now, if GK, AD be drawn ⊥r to BC meeting it in K, D respectively, then from similar Δs,

$$\frac{A'K}{GK} = \frac{A'D}{AD}$$

or

$$\frac{BK - KC}{GK} = \frac{BD - CD}{AD}.$$

Hence $\quad \cot GBC - \cot GCB = \cot B - \cot C.$

$$\therefore \quad \Sigma \cot GAB = \Sigma \cot GBA$$

$$= \tfrac{1}{2}\left\{\frac{BK + KC}{GK} + \ldots + \ldots\right\}$$

$$= \tfrac{1}{2}\left\{\frac{3a}{h_1} + \frac{3b}{h_2} + \frac{3c}{h_3}\right\},\ h_1 = AD, \text{ etc.}$$

$$= 3 \cdot \frac{a^2 + b^2 + c^2}{4\Delta}$$

$$= 3 \cot \omega.$$

Hence the result follows.

Cor.—If K be the symmedian point,

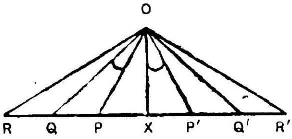
$$\cot KAB \cot KBC \cot KCA + \cot KAC \cot KCB \cot KBA = 6 \cot \omega.$$

Question 1233.

(G. V. T.) :—P, P′ ; Q, Q′ ; R, R′ are opposite vertices of a complete quadrilateral circumscribing a circle concentric with the in-circle of a triangle. If P, P′ are isogonal conjugates with respect to the triangle, show that Q, Q′ and R, R′ are also isogonal conjugates.

Solution by K. Srinivasaraghavan.

Lemma :—If in an involution two pairs of lines are symmetrically placed about an axis then all the pairs of lines are also symmetrically placed.



Let OX be the axis and OP, OP′ ; OQ, OQ′ are symmetrically placed and the pencil O(RQPP′Q′R′) is an involution pencil. Take the st. line ⊥ to the axis and let the involution range be RQPP′Q′R′.

Now $QP = Q'P'$ and $PX = P'X$.

Again $(RQPP') = (R'Q'P'P)$

i.e. $$\frac{RQ \,.\, PP'}{QP \,.\, RP'} = \frac{R'Q' \,.\, PP'}{Q'P' \,.\, R'P}.$$

i.e. $$\frac{RQ}{RP'} = \frac{R'Q'}{R'P}.$$

$\therefore$ $$\frac{QP'}{RQ} = \frac{Q'P}{R'Q'}.$$

But $$QP' = Q'P.$$

$$\therefore \quad RQ = R'Q'.$$

Hence OR and OR′ are symmetrically placed about OX. All the other pairs are also like this.

Let the triangle be ABC, and let PP′QQ′RR′ be the complete quadrilateral—P, P′ being isogonal conjugate points. Let AT, AT′ be the tangents from A to the circle inscribed in the quadrilateral.

Now AT, AT′; AP, AP′; AQ, AQ′; AR, AR′ form the same involution.

Also AT and AT′ are equally inclined to AB and AC, since they are tangents from A to two concentric circles. AI is the axis about which AT, AT′; AB, AC; AP, AP′ are symmetrically placed.

Thus in the involution AP, AP′; AQ, AQ′; AR, AR′; AT, AT′; the rays AP, AP′; AT, AT′ are symmetrically placed about AI.

Hence AQ, AQ′; AR, AR′ are also symmetrically placed about AI; that is, they are equally inclined to AB and AC. Similarly it can be proved that QB and Q′B as well as RB and R′B are isogonal conjugate lines, whence the result.

Question 1236.

(VIDWAN G. SOMANNA, M.A., L.T.):—The sum of four different weights is 40 seers. The weights can be manipulated to deal out any quantity weighing up to 40 seers, in integral seers. Find the weights.

Solution by Mrs. Edith Thomas, N. Maitra, K. Srinivasaragharan, P. R. Venkatakrishna Iyer and S. Audinarayanan.

Performing successive divisions by 3 on 40,

```
3 | 40
3 | 13 — 1
 3 | 4 — 1
     1 — 1,
```

We find the following numbers

$$1,\ 1 \times 3,\ 1 \times 3 \times 3, \text{ and } 1 \times 3 \times 3 \times 3,$$

These are 1, 3, 9, 27 (the sum of which is 40); all weights from 1 up to 40 seers are obtained by taking positively or negatively one or more numbers from among the set 1, 3, 9, 27; for instance, $22 = 27 + 3 + 1 - 9$.

The required weights are thus 1, 3, 9 and 27 seers.

Question 1237.

(VIDWAN G. SOMANNA, M.A., L.T.):—The aggregate value of ten different coins is Rs. 360. If they can be manipulated to pay any sum (in integral rupees) up to Rs. 360, find the values of the coins.

Solution by K. Srinivasaraghavan and N. Maitra.

Since money is to be given, we must get a series such that the sum of any number of terms must give natural numbers. Such a series is

1, 2, 4, 8, 16, 32, 64, 128, 256, 512, ...

The sum of the first 8 terms is 255. Since we want two more coins to make up 105, they can be selected (excluding 1, 2, 4, 8......128) in 45 different ways as [(3, 102), (5, 100), (6, 99)......].

Thus there are 45 ways of having the required 10 coins.

Question 1241.

(A. C. L. WILKINSON):—Solve the differential equation

$$2a^2xy^2\frac{d^2y}{dx^2} + 1 = 0.$$

[A special form of this equation $a = \frac{2}{3}$ is given in Forsyth: *Differential Equations*, 3rd Edition, Misc. Examples, 51 (ii); but in his solutions published in 1918, Forsyth states that he has not been able to obtain a solution in a finite form. A solution in a finite form exists.]

Remarks by Prof. N. M. Shah.

Principal Wilkinson has given the required solution of this question while solving a particular case. (See his solution of Question 1152 published in the August number of this Journal).

Question 1247.

(N. B. MITRA) :—If $p\ (= 2n+1)$ be a prime, and if a, b, c be the digits in the unit's place in

$$\frac{2n!+1}{p},\quad \frac{n!+(-1)^n}{p},\quad \frac{2^{4n}-1}{p},$$

when these are expressed in the scale of p, prove that

$$a-b+c \equiv 0 \ (\text{mod.}\ p).$$

[*N.B.*—The 2nd number is evidently $\dfrac{(n!)^2+(-1)^n}{p}$; for, $n!+(-1)^n$ is not a multiple of p].

Solution by I. Totadri Iyengar.

$$\left.\begin{aligned} \text{Let}\quad \frac{2n!+1}{p} &= a + a_1 p + a_2 p^2 + \ldots\ldots, \\ \frac{(n!)^2+(-1)^n}{p} &= b + b_1 p + b_2 p^2 + \ldots\ldots, \\ \frac{2^{4n}-1}{p} &= c + c_1 p + c_2 p^2 + \ldots\ldots. \end{aligned}\right\} \quad \ldots \text{(A)}$$

$$\text{Now } 2n! = \left\{\frac{p-(p-2)}{2}\right\}\left\{\frac{p-(p-4)}{2}\right\}\ldots\ldots\frac{p+1}{2}\cdot\frac{p-1}{2}$$

$$\ldots\ldots\left\{\frac{p+p-4}{2}\right\}\left\{\frac{p+p-2}{2}\right\}$$

$$= \frac{1}{2^{2n}}\{p^2-1^2\}\{p^2-3^2\}\ldots\ldots\{p^2-(p-2)^2\}.$$

$$\therefore\quad 2^{2n}.2n! \equiv (-1)^n\, 1^2.3^2\ldots(p-2)^2 \ [\text{mod. } p^2]$$

$$2^{2n}.2n!\ 2^{2n}(n!)^2 \equiv (-1)^n\, 1^2.2^2.3^2\ldots\ldots(p-2)^2(p-1)^2 \ [\text{mod. } p^2]$$

$$\equiv (-1)^n (2n!)^2.$$

$$\therefore\quad 2n! \equiv (-1)^n\, 2^{4n}(n!)^2. \quad \ldots \quad \ldots \quad \ldots \text{(B)}$$

$$\text{Now}\quad 2n! \equiv -1+ap$$

$$(n!)^2 \equiv -(-1)^n + bp$$

$$2^{4n} \equiv 1 + cp.$$

Substituting in (B), we get

$$-1+ap \equiv (-1)^n\{1+cp\}\{bp-(-1)^n\}.$$

$$\equiv (-1)^n\{bp-(-1)^n-(-1)^n cp\}$$

$$\equiv (-1)^n bp - 1 - cp.$$

$$\therefore\quad a-(-1)^n b + c \equiv 0 \ (\text{mod. } p).$$

N.B. $a-b+c \equiv 0$ (mod. p) is a mis-statement, as can easily be seen by taking a particular case and verifying.

Question 1249.

(K. C. SHAH, M.A.) :—Through a fixed point P in the plane of a triangle ABC, a variable st. line is drawn cutting the sides BC, CA, AB in A′, B′, C′ respectively. If any point Q is taken on this straight line such that

$$\frac{1}{A'Q}+\frac{1}{B'Q}+\frac{1}{C'Q}=\frac{3}{PQ},$$

prove that the locus of Q is a conic section. Locate the position of P for the different species of the conic.

Solutions: (*1*) *by C. Ranganathan and* (2) *by K. Satyanarayana.*

(1) Taking the given triangle as the triangle of reference, let the trilinear co-ordinates of the point P be (x', y', z').

Any line through (x', y', z') is given by

$$\frac{x-x'}{l}=\frac{y-y'}{m}=\frac{z-z'}{n}=r,$$

where r is the distance between (x, y, z) and (x', y', z').

This line cuts BC at the point given by $x=0$

$$\therefore PA'=-\frac{x'}{l}. \quad \therefore \; QA'=PA'-PQ=-\frac{x'}{l}-\frac{x-x'}{l}=-\frac{x}{l}.$$

$$\therefore \quad \frac{1}{QA'}+\frac{1}{QB'}+\frac{1}{QC'}=-\left(\frac{l}{x}+\frac{m}{y}+\frac{n}{z}\right).$$

Hence
$$-\left(\frac{l}{x}+\frac{m}{y}+\frac{n}{z}\right)=-\frac{3}{r}.$$

$$\therefore \quad \frac{x-x'}{x}+\frac{y-y'}{y}+\frac{z-z'}{z}-3=0,$$

since
$$x-x'=lr, \text{ etc.}$$

Therefore
$$\frac{x'}{x}+\frac{y'}{y}+\frac{z'}{z}=0$$

is the locus of Q and this conic is circumscribed to the triangle of reference.

The position of the point P for different species of the conic can be determined by applying the conditions for an ellipse, a hyperbola, a rectangular hyperbola or a parabola.

(2) Let BC, and the altitude from A be the axes, and let A, B, C be $(0, a)$, $(b, 0)$, $(c, 0)$, P being (h, k).

Then AB, BC, CA are $\frac{x}{b}+\frac{y}{a}=1$; $y=0$; $\frac{x}{c}+\frac{y}{a}=1$.

Points on any line through P, as usual, are

$$x = h + r\cos\theta,\ y = k + r\sin\theta.$$

If 'r' be taken as r_1, r_2, r_3, r for A_1, B_1, C_1, Q; the condition becomes

$$\Sigma \frac{1}{r - r_1} = \frac{3}{r}. \qquad \ldots \qquad \ldots \qquad \ldots \quad (1)$$

It is easily seen that

$$\frac{r}{r_1} = -\frac{r\sin\theta}{k};\ \frac{r}{r_2} = \frac{\frac{r\cos\theta}{c} + \frac{r\sin\theta}{a}}{1 - \frac{h}{c} - \frac{k}{a}};\ \frac{r}{r_3} = \frac{\frac{r\cos\theta}{b} + \frac{r\sin\theta}{a}}{1 - \frac{h}{b} - \frac{k}{a}}$$

∴ Q being the point (x, y), (1) becomes

$$\frac{k}{y} + \frac{\frac{h}{c} + \frac{k}{a} - 1}{\frac{x}{c} + \frac{y}{a} - 1} + \frac{\frac{h}{b} + \frac{k}{a} - 1}{\frac{x}{b} + \frac{y}{a} - 1} = 0. \qquad \ldots \quad (2)$$

which is a conic. On simplification, the second degree terms reduce to

$$\frac{1}{a^2bc}\left[ka^2x^2 + 2xy\left\{ka(b+c) + a^2\left(h - \frac{b+c}{2}\right)\right\}\right.$$
$$\left. + y^2\{3kbc + ha(b+c) - 2abc\}\right].$$

Hence (2) is an ellipse, parabola, or hyperbola, according as

$$\left[k(b+c) + a\left(h - \frac{b+c}{2}\right)\right]^2$$
$$- k\left[3kbc + ha(b+c) - 2abc\right] \lesseqgtr 0. \qquad \ldots \quad (3)$$

Putting x and y for $h - \dfrac{b+c}{2}$ and k, (3) becomes

$$a^2x^2 + a(b+c)xy + y^2(b^2 + c^2 - bc) - \frac{a(b-c)^2}{2}y \lesseqgtr 0 \quad (3')$$

Of course x, y are the co-ordinates of P with respect to parallel axes through mid. pt. of BC.

But $\quad a^2x^2 + a(b+c)xy + y^2(b^2 + c^2 - bc) - \dfrac{a(b-c)^2}{2}y = 0$

is an ellipse touching the x-axis at the origin, *i.e.*, touching BC at its mid. pt. The ellipse therefore touches the sides at the middle points.

Thus (2) is an ellipse, parabola or hyperbola according as P lies within, on, or without this ellipse.

Additional Solution by P. R. Venkatakrishna Iyer.

Questions for Solution.

1296. (V. RAMASWAMY AIYAR) :—Prove that, for every acute-angled triangle ABC, there is an obtuse-angled triangle A′B′C′, and *vice versa*, such that if O, O′ be the ortho-centres of these triangles, we shall have

$$OA = O'A',\ OB = O'B',\ OC = O'C'.$$

If the two triangles be placed so that their circum-centres coincide, show that the circum-circle of either will touch the nine-points-circle of the other.

1297. (F. H. V. GULASEKHARAM):—Three circles of centres A, B, C and curvatures c_1, c_2, c_3 respectively touch each other externally and are all touched by a fourth circle (*not including these*) of centre D and curvature c_4. Show that the incircles of the triangles DBC, DCA, DAB touch each other externally and are all touched by the incircle of the triangle ABC.

If $$d_1,\ d_2,\ d_3,\ d_4$$
be the curvatures of these incircles, prove the following relations :—

(i) $d_1^2 = c_2 c_3 + c_3 c_4 + c_4 c_2$; etc.

$\quad c_1^2 = d_2 d_3 + d_3 d_4 + d_4 d_2$; etc.

(ii) $d_1 + d_2 + d_3 + d_4 = c_1 + c_2 + c_3 + c_4$

$$= 2(d_1 + c_1) \quad = 2(d_2 + c_2)$$
$$= 2(d_3 + c_3) \quad = 2(d_4 + c_4).$$

Deduce that

$$c_1^2 + c_2^2 + c_3^2 + c_4^2$$
$$= 2[c_2 c_3 + c_3 c_1 + c_1 c_2 + c_4 c_1 + c_4 c_2 + c_4 c_3]$$

and obtain solutions to Questions 759 and 1091, (*J. I. M. S.*)

What modifications are necessary when the circle (D, c_4) *includes* the circles (A, c_1), (B, c_2), (C, c_3) ?

1298. (I. TOTADRI IYENGAR) :—The function $\phi(n)$ denotes the number of numbers less than n and prime to it, n being any positive integer. The function $\psi(x)$ stands for the infinite series

$$\phi(1)\, x - \phi(3)\, x^3 + \phi(5)\, x^5 - \phi(7)\, x^7 + \ldots\ldots\ldots$$

Show that the infinite series

$$\psi(\tan\theta) - \psi(\tan^3\theta) + \psi(\tan^5\theta) - \psi(\tan^7\theta) + \ldots\ldots\ldots$$
$$= \frac{\sin 4\theta}{4}.$$

1299. (I. TOTADRI IYENGAR):—Let the no. of combinations of n things r at a time be symbolised by $\binom{n}{r}$.

Prove that

$$\binom{n}{1} + 5\binom{n}{3} + 5^2\binom{n}{5} + 5^3\binom{n}{7} + \ldots\ldots$$

$$= 2^{n-1} + 2^2 \Big[\Big\{ \binom{n-2}{1} + \binom{n-3}{1} + \binom{n-4}{1} + \ldots\ldots \Big\}$$

$$+ 5 \Big\{ \binom{n-2}{3} + \binom{n-3}{3} + \binom{n-4}{3} + \ldots\ldots \Big\}$$

$$+ 5^2 \Big\{ \binom{n-2}{5} + \binom{n-3}{5} + \binom{n-4}{5} + \ldots\ldots \Big\}$$

$$+ \ldots\ldots\ldots\ldots \quad + \ldots\ldots\ldots\ldots \Big].$$

1300. (M. G. INAMDAR):—Prove that the area of a triangle can be got from the formula

$$\Delta = 4 \{ (\sigma - p)(\sigma - q)(\sigma - r)(\sigma - R) \}^{\frac{1}{2}}$$

where $2\sigma = (p + q + r + R)$; p, q, r are the perpendiculars from the circum-centre on the sides of the triangle; and R is the circum-radius.

1301. (S. RAJANARAYANAN):—If n sets of quantities $(a_0, a_1, \ldots\ldots)$, $(b_0, b_1, b_2 \ldots\ldots) \ldots\ldots\ldots (l_0, l_1 \ldots\ldots)$ are related in the following manner :—

$$a_{r+1}^2 - 1 = b_r, \; b_{r+1}^2 - 2 = c_r, \; c_{r+1}^2 - 3 = d_r, \ldots\ldots\ldots$$

sum the series

$$1 + \frac{b_1}{a_2 + a_1} + \frac{c_1}{(a_3 + a_2)(b_2 + b_1)} + \frac{d_1}{(a_4 + a_3)(b_3 + b_2)(c_2 + c_1)} +$$

$$\ldots\ldots\ldots + \frac{l_1}{(a_n + a_{n-1})(b_{n-1} + b_{n-2})\ldots\ldots\ldots}$$

1302. (S. RAJANARAYANAN):—ABC is any triangle. D E F are points on the sides such that AD, BE, CF are concurrent. If EF, FD meet BC, CA in G and H respectively, prove that

$$EF.\, FD.\, CG.\, AH = GF.\, HF.\, CD.\, AE$$

195317

LIST OF JOURNALS RECEIVED BY THE SOCIETY.

1 Messenger of Mathematics
2 Quarterly Journal of Mathematics
3 Mathematical Gazette
4 The Annals of Mathematics
5 American Journal of Mathematics
6 Bulletin of the American Mathematical Society
7 Transactions of the American Mathematical Society
8 Monthly Notices of the Royal Astronomical Society
9 Proceedings of the Royal Society of London
10 The Philosophical Magazine and Journal of Science
11 Astrophysical Journal
12 Crelle's Journal
13 L'intermediaere des Mathematiciens
14 Mathematische Annalen
15 Philosophical Transactions of the Royal Society of London
16 Acta Mathematica
17 Popular Astronomy
18 Proceedings of the Edinburgh Mathematical Society
19 Proceedings of the London Mathematical Society
20 Mathematics Teacher
21 Bulletin of the Calcutta Mathematical Society
22 The Tohoku Mathematical Journal
23 Nature
24 The American Mathematical Monthly
25 Proceedings of the Benares Mathematical Society

The Indian Mathematical Society

(Founded in 1907 for the Advancement of Mathematical Study and Research in India.)

THE COMMITTEE.

President:

BALAK RAM, M.A, I.C.S., Judge, Bijapur.

Treasurer:

S. NARAYANA AIYAR, M.A., Chief Accountant, Port Trust Office, Madras.

Librarian:

V. B. NAIK, M.A., Professor of Mathematics, Fergusson College, Poona.

Secretaries:

P. V. SESHU AIYAR, B.A., L.T., Professor of Applied Mathematics, Presidency College, Madras.

M. T. NARANIENGAR, M.A., Professor of Mathematics, Central College, Bangalore.

Additional Members:

The Hon'ble R. P. PARANJPYE, M.A., D.Sc., Minister of Education, Bombay.

E. B. ROSS, M.A., Professor of Mathematics, Christian College, Madras.

D. D. KAPADIA, M.A., B.Sc., Professor of Mathematics, Elphinstone College, Bombay.

HEMRAJ, M.A., Professor of Mathematics, Dayal Singh College, Lahore.

K. ANANDA RAO, M.A., Professor of Pure Mathematics, Presidency College, Madras.

S. V. RAMAMURTHY, M.A., I.C.S., Ramnad.

A. C. L. WILKINSON, M.A., Principal and Professor of Mathematics, Deccan College, Poona.

OFFICERS OUTSIDE THE COMMITTEE.

Assistant Secretary:

N. RAGHUNATHA AIYANGAR, M.A., Assistant Professor of Mathematics, Presidency College, Madras.

Assistant Librarian:

V. A. APTE, M.A., Professor of Mathematics, Fergusson College, Poona.

Associates:

K. B. MADHAVA, M.A.
G. A. SRINIVASAN, M.A., L.T.
A. NARASINGA ROW, M.A., L.T.
S. R. RANGANATHAN, M.A., L.T.

Head-Quarters: Poona.